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Fluctuating hydrodynamics for driven granular gases

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Abstract. We study a granular gas heated by a stochastic thermostat in the dilute limit. Starting from the kinetic equations governing the evolution of the correlation functions, a Boltzmann-Langevin equation is constructed. The spectrum of the corresponding linearized Boltzmann-Fokker-Planck operator is analyzed, and the equation for the fluctuating transverse velocity is derived in the hydrodynamic limit. The noise term (Langevin force) is thus known microscopically and contains two terms: one coming from the thermostat and the other from the fluctuating pressure tensor. At variance with the free cooling situation, the noise is found to be white and its amplitude is evaluated.

1 Introduction

Typically, a granular system is defined as an ensemble of macroscopic particles which collide inelastically, i.e. part of the kinetic energy of the grains is dissipated in a collision. This simple ingredient gives rise to a very rich phenomenology which is of interest not only from practical or industrial perspective, but also because of the resulting new theoretical challenges [1–4]. One of the most widely employed idealized model for granular fluids is a system of smooth hard spheres (or disks in two dimensions) whose collisions are characterized by a constant coefficient of normal restitution [5,6]. For this model, and considering that the particles move freely between collisions, kinetic equations have been derived: starting from the dynamics of the particles, it is possible to derive the corresponding Liouville equation, and the Boltzmann equation results in the low density limit [7,8]. This kinetic equation has been extensively used to address many fundamental questions such as the derivation of the hydrodynamic equations, with explicit expressions for the transport coefficients, which have been derived by the Chapman-Enskog method [9,10] and also via the linearized Boltzmann equation [11]. Due to the inelasticity of the collisions, the total energy of an isolated granular system decays monotonically in time. In the fast-flow regime, it has been shown numerically that, for a wide class of initial conditions, the system reaches the so-called *Homogeneous Cooling State* (HCS), in which all the time dependence of the one-particle distribution function goes through the granular temperature, which is defined as the second velocity moment of the distribution [12,13]. This state has been extensively studied in the literature and very recently the fluctuations of the transverse velocity have been analyzed [14,15]. It has been found that the transverse velocity fulfills a Langevin equation but, in contrast to the elastic case, the noise is not white and the second moment of the fluctuations is not only controlled by the viscosity but also depends on a new coefficient. Similar results are found for the other hydrodynamic equations [16]. The study of fluctuations in the HCS is important for the development of a general theory of fluctuations in granular systems because it defines the reference state from which macroscopic hydrodynamic equations can be derived [9]. In this sense, the HCS plays, for inelastic gases, a role similar to the equilibrium state for molecular gases.

On the other hand, there are situations in which the grains cannot be considered to move freely between collisions. If, for example, the grains are immersed in a medium which acts as a thermostat, the system may reach a stationary state in which the energy injected by the thermostat is compensated by the energy dissipated in collisions. Note that, if the grains are Brownian particles, the interstitial medium injects energy into the granular system, but also acts as an energy sink due to frictional forces. One of the simplest mechanism that can be considered to thermalize the system is a white noise force acting on each grain, which results in the so-called stochastic thermostat [17–28]. One important point is that the distribution function differs from that of the HCS [17], and that it is this distribution which plays the role of the "reference state". The non-equilibrium steady state that the system reaches in the long time limit exhibits long-range correlations which are in agreement with the predictions of fluctuating hydrodynamics [20]. The latter description was introduced phenomenologically, and is expected to be valid in the vicinity of the elastic limit only. The objective of this work is to derive these equations from a more fundamental point of view and without the restriction of small inelasticity. More precisely, we adapt the formalism worked out in [15] for the free cooling, to the present driven case. Starting from a Boltzmann-Langevin description, we derive a fluctuating equation for the transverse velocity identifying the noise of this equation. Under certain hypothesis to be clarified in the text, we obtain that the correlation function of the noise is well approximated by the one introduced in Ref. [20], where the internal noise contribution (excluding the "external" noise term directly stemming from the thermostat) fulfilled a fluctuation-dissipation relation as for conservative fluids [29].

The remainder of the paper is organized as follows. In Section 2 previous results for a system heated by a stochastic thermostat are presented, such as the equations for one-particle distribution function and the two-particle correlation function. In Section 3 the Boltzmann-Langevin equation for this system is derived and the properties of the noise are inferred. The particular case of the transverse velocity field is analyzed in Section 4 and finally, the conclusions are presented in Section 5

2 Stochastic thermostat: Preliminary results

The system considered is a dilute gas of N smooth inelastic hard particles of mass m and diameter σ. The position and velocity of the ith particle at time t will be denoted by $\mathbf{R}_i(t)$ and $\mathbf{V}_i(t)$, respectively. The effect of a collision between two particles i and j is to instantaneously modify their velocities according to the collision rule

$$\mathbf{V}_i' = \mathbf{V}_i - \frac{1+\alpha}{2}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{V}_{ij})\hat{\boldsymbol{\sigma}}\,, \qquad \mathbf{V}_j' = \mathbf{V}_j + \frac{1+\alpha}{2}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{V}_{ij})\hat{\boldsymbol{\sigma}}\,, \tag{1}$$

where $\mathbf{V}_{ij} \equiv \mathbf{V}_i - \mathbf{V}_j$ is the relative velocity, $\hat{\boldsymbol{\sigma}}$ is the unit vector pointing from the center of particle j to the center of particle i at contact, and α is the coefficient of normal restitution. It is defined in the interval $0 < \alpha \leq 1$ and it will be considered here as a constant, independent of the relative velocity. Between collisions, the system is heated uniformly by adding a random velocity to the velocity of each particle at equal times. The driving is implemented in such a way that the time between random kicks is small compared to the mean free time. Then, between collisions, the velocities of the particles undergo a large number of kicks due to the thermostat. In addition, we will assume that the "jump moments" of the velocities of the particles verify

$$B_{ij,\beta\gamma} \equiv \lim_{\Delta t \to 0} \frac{\langle \Delta V_{i,\beta} \Delta V_{j,\gamma} \rangle_{noise}}{\Delta t} = \xi_0^2 \delta_{ij}\delta_{\beta\gamma} + \frac{\xi_0^2}{N}(\delta_{ij} - 1)\delta_{\beta\gamma}, \tag{2}$$
$$i,j = 1,\ldots,N \qquad \text{and} \qquad \beta,\gamma = 1,\ldots,d$$

where we have introduced $\Delta V_{i,\beta} \equiv V_{i,\beta}(t+\Delta t) - V_{i,\beta}(t)$, $V_{i,\beta}(t)$ being the β component of the velocity of particle i at time t. We have also introduced the strength of the noise, ξ_0^2, and $\langle \ldots \rangle_{noise}$, which denotes average over different realizations of the noise. The non-diagonal terms (corresponding to $i \neq j$ and $\beta = \gamma$) are necessary in order to conserve the total momentum [32].

2.1 Kinetic equations

Given a trajectory of the system, one-point and two-point microscopic densities in phase space at time t are defined by

$$F_1(x_1,t) = \sum_{i=1}^{N} \delta[x_1 - X_i(t)], \tag{3}$$

and

$$F_2(x_1,x_2,t) = \sum_{i=1}^{N}\sum_{j\neq i}^{N} \delta[x_1 - X_i(t)]\delta[x_2 - X_j(t)], \tag{4}$$

respectively. Here $X_i(t) \equiv \{\mathbf{R}_i(t), \mathbf{V}_i(t)\}$, while the $x_i \equiv \{\mathbf{r}_i, \mathbf{v}_i\}$ are field variables referring to the one-particle phase space. The average of $F_1(x_1,t)$ and $F_2(x_1,x_2,t)$ over the initial probability distribution of the system $\rho(\Gamma,0)$, where $\Gamma \equiv \{X_1(t),\ldots,X_N(t)\}$, are the usual one-particle and two-particle distribution functions,

$$f_1(x_1,t) = \langle F_1(x_1,t)\rangle, \qquad f_2(x_1,x_2,t) = \langle F_2(x_1,x_2,t)\rangle, \tag{5}$$

with the notation

$$\langle G\rangle \equiv \int \mathrm{d}\Gamma G(\Gamma)\rho(\Gamma,0). \tag{6}$$

In the dilute limit, assuming molecular chaos, i.e. that no correlations exist between colliding particles, and that the sizes of the jumps due to the thermostat are small compared to the scale on which the distribution varies, the equation for the one-particle distribution function, $f_1(x_1,t)$, is the Boltzmann-Fokker-Planck equation [17]

$$\frac{\partial}{\partial t} f_1(x_1,t) + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1} f_1(x_1,t) = J[f_1|f_1] + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial \mathbf{v}_1^2} f_1(x_1,t), \tag{7}$$

where

$$J[f_1|f_1] = \int \mathrm{d}x_2 \delta(\mathbf{r}_{12})\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2) f_1(x_1,t) f_1(x_2,t) \tag{8}$$

and

$$\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2) = \sigma^{d-1}\int \mathrm{d}\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}_{12})[\alpha^{-2}b_{\hat{\boldsymbol{\sigma}}}^{-1} - 1], \tag{9}$$

is the binary collision operator. The operator $b_{\hat{\boldsymbol{\sigma}}}^{-1}$ changes the velocities to its right into the pre-collisional velocities

$$\mathbf{v}_1^* = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}_{12})\hat{\boldsymbol{\sigma}}\,, \qquad \mathbf{v}_2^* = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}_{12})\hat{\boldsymbol{\sigma}}\,. \tag{10}$$

As can be seen, the last term in (7) does not appear in the free cooling case and depends on the strength of the heating, ξ_0.

Let us introduce the two-particle correlation function through the usual cluster expansion

$$f_2(x_1,x_2,t) = f_1(x_1,t)f_1(x_2,t) + g_2(x_1,x_2,t). \tag{11}$$

Neglecting three-body correlations, the equation for the correlation function $g_2(x_1,x_2,t)$ was derived in [32] and reads

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{r}_2}\right] g_2(x_1,x_2,t)\\
&\quad = \delta(\mathbf{r}_{12})\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2) f_1(x_1,t)f_1(x_2,t) + [K(x_1,t)+K(x_2,t)]g_2(x_1,x_2,t)\\
&\qquad -\frac{\xi_0^2}{N}\frac{\partial}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{v}_2} f_1(x_1,t)f_1(x_2,t),
\end{aligned} \tag{12}$$

where

$$K(x_i,t) = \int \mathrm{d}x_3 \delta(\mathbf{r}_{i3})\bar{T}_0(\mathbf{v}_i,\mathbf{v}_3)(1+\mathcal{P}_{i3}) f_1(x_3,t) + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial \mathbf{v}_i^2}, \tag{13}$$

with $\mathcal{P}_{ab}$ an operator that interchanges the label a and b in the quantities to its right.

2.2 The stationary state

It has been shown numerically that, for a wide class of initial conditions, the system reaches a homogeneous stationary state [20] in which the energy input from the thermostat is compensated by the energy lost in collisions. In this case the Boltzmann-Fokker-Planck equation reads

$$\frac{\xi_0^2}{2}\frac{\partial^2}{\partial \mathbf{v}_1^2} f_H(\mathbf{v}_1) + J[f_H|f_H] = 0. \tag{14}$$

For the sake of simplicity, let us introduce the following dimensionless distribution

$$f_H(\mathbf{v}_1) = \frac{n_H}{v_H^d}\chi_H(c), \tag{15}$$

with $\chi_H(c)$ an isotropic function of the modulus of $\mathbf{c}$, $v_H = (\frac{2T_H}{m})^{1/2}$, $\mathbf{c} = \frac{\mathbf{v}}{v_H}$ and T_H the granular temperature defined as

$$\frac{d}{2} n_H T_H = \int d\mathbf{v} \frac{1}{2} m v^2 f_H(\mathbf{v}). \tag{16}$$

By introducing equation (15) in equation (14), we obtain a closed equation for χ_H

$$\int d\mathbf{c}_2 \bar{T}(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) + \frac{\tilde{\xi}^2}{2}\frac{\partial}{\partial \mathbf{c}_1^2}\chi_H(c_1) = 0, \tag{17}$$

where

$$\bar{T}(\mathbf{c}_1, \mathbf{c}_2) = \int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})[\alpha^{-2}b_{\hat{\boldsymbol{\sigma}}}^{-1} - 1], \tag{18}$$

is the dimensionless binary collision operator and $\tilde{\xi}^2 = \frac{\xi_0^2 \ell}{v_H^3}$ is the dimensionless strength of the noise with $\ell = (n_H\sigma^{d-1})^{-1}$ proportional to the mean free path. In the case of the correlation function, it is convenient to introduce its dimensionless counterpart, $\tilde{g}_H$, as

$$g_{2,H}(x_1, x_2) = \frac{n_H}{\ell^2 v_H^{2d}}\tilde{g}_H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2), \tag{19}$$

where we have introduced the dimensionless length scale $\mathbf{l} = \mathbf{r}/\ell$ and $\mathbf{l}_{12} = \mathbf{l}_1 - \mathbf{l}_2$. The reduced distribution fulfills

$$\begin{aligned}&\left[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2) - \mathbf{c}_{12}\cdot\frac{\partial}{\partial \mathbf{l}_{12}}\right]\tilde{g}_H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2)\\ &\quad = -\delta(\mathbf{l}_{12})\bar{T}(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) + \tilde{\xi}^2\frac{n_H\ell^d}{N}\frac{\partial}{\partial \mathbf{c}_1}\cdot\frac{\partial}{\partial \mathbf{c}_2}\chi_H(c_1)\chi_H(c_2),\end{aligned} \tag{20}$$

where $\Lambda(\mathbf{c}_i)$ is the linearized Boltzmann-Fokker-Planck operator

$$\Lambda(\mathbf{c}_i)h(\mathbf{c}_i) = \int d\mathbf{c}_3 \bar{T}(\mathbf{c}_i, \mathbf{c}_3)(1 + \mathcal{P}_{i3})\chi_H(c_3)h(\mathbf{c}_i) + \frac{\tilde{\xi}^2}{2}\frac{\partial^2}{\partial \mathbf{c}_i^2}h(\mathbf{c}_i). \tag{21}$$

Equation (20) describes the one-time correlation between fluctuations in the stationary state. As it can be seen, the correlation function is determined by the properties of the linearized Boltzmann-Fokker-Planck operator, Λ, and by the one-particle distribution function, χ_H.

For the purpose of the next section, it is also convenient to define a new function

$$h_H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2) \equiv \chi_H(c_1)\delta(\mathbf{l}_{12})\delta(\mathbf{c}_{12}) + \tilde{g}_H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2). \tag{22}$$

Taking into account equation (20) and that the term $\mathbf{c}_{12} \cdot \frac{\partial}{\partial \mathbf{l}_{12}} \chi_H(c_1)\delta(\mathbf{l}_{12})\delta(\mathbf{c}_{12})$ vanishes identically, the equation for this quantity is

$$\left[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2) - \mathbf{c}_{12} \cdot \frac{\partial}{\partial \mathbf{l}_{12}}\right] h_H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2)$$
$$= -\delta(\mathbf{l}_{12})\Gamma(\mathbf{c}_1, \mathbf{c}_2) + \tilde{\xi}^2 \frac{n_H \ell^d}{N} \frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{\partial}{\partial \mathbf{c}_2} \chi_H(c_1)\chi_H(c_2), \tag{23}$$

where

$$\Gamma(\mathbf{c}_1, \mathbf{c}_2) = \bar{T}(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) - [\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2)]\chi_H(c_1)\delta(\mathbf{c}_{12}). \tag{24}$$

2.3 Linearized Boltzmann-Fokker-Planck equation

In this section we derive the evolution equation for a small perturbation around the homogeneous stationary distribution function $f_H(\mathbf{v})$. The objective is to evaluate some spectral properties of the linear operator that controls the dynamics, which turns out to be the linearized Boltzmann-Fokker-Planck operator introduced above. These properties will be useful in the last section where we derive hydrodynamic equations.

Let us introduce the small perturbation, δf

$$\delta f(x_1, t) = f_1(x_1, t) - f_H(\mathbf{v}_1), \qquad \frac{\delta f}{f_1} \ll 1. \tag{25}$$

The linearized equation for δf is obtained from the Boltzmann-Fokker-Planck equation (7) and equation (14)

$$\frac{\partial}{\partial t}\delta f(x_1, t) + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1}\delta f(x_1, t) = K(x_1, t)\delta f(x_1, t), \tag{26}$$

where $K(x_1, t)$ is defined in (13). Now, let us introduce a dimensionless perturbation and a dimensionless time scale as

$$\delta f(x_1, t) = \frac{n_H}{v_H^d}\delta\chi(\mathbf{l}, \mathbf{c}, s), \tag{27}$$

$$s = \frac{v_H}{\ell} t, \tag{28}$$

which is proportional to the number of collisions per particle in the interval $(0, t)$. The equation for $\delta\chi$ reads

$$\frac{\partial}{\partial s}\delta\chi(\mathbf{l}, \mathbf{c}_1, s) = \left[\Lambda(\mathbf{c}_1) - \mathbf{c}_1 \cdot \frac{\partial}{\partial \mathbf{l}}\right] \delta\chi(\mathbf{l}, \mathbf{c}_1, s), \tag{29}$$

where Λ is the linearized Boltzmann-Fokker-Planck operator defined in (21). Equation (29) is the linearized Boltzmann-Fokker-Planck equation and it describes the dynamics of any small perturbation around the homogeneous stationary state.

In [32] it was shown that the linearized Boltzmann-Fokker-Planck operator has $d+1$ eigenfunctions associated to the null eigenvalue (in principle there are not more eigenfunctions associated to this eigenvalue). These eigenfunctions are

$$\xi_1(\mathbf{c}) = \frac{1}{3}\frac{\partial}{\partial \mathbf{c}} \cdot [\mathbf{c}\chi_H(c)] + \chi_H(c), \tag{30}$$

$$\boldsymbol{\xi}_2(\mathbf{c}) = -\frac{\partial}{\partial \mathbf{c}}\chi_H(c). \tag{31}$$

They fulfill $\Lambda(\mathbf{c})\xi_i(\mathbf{c}) = 0$ for $i = 1, 2$. Moreover, as the number of particles and momentum are conserved in collisions, we have

$$\int d\mathbf{c}\Lambda(\mathbf{c})h(\mathbf{c}) = \int d\mathbf{c}c_i\Lambda(\mathbf{c})h(\mathbf{c}) = 0, \tag{32}$$

where h is an arbitrary function. Equivalently, it can be said that the functions

$$\bar{\xi}_1(\mathbf{c}) = \chi_H(c), \qquad \bar{\boldsymbol{\xi}}_2(\mathbf{c}) = \chi_H(c)\mathbf{c}, \tag{33}$$

are left eigenfunctions of Λ associated to the null eigenvalue with the scalar product

$$\langle f|g\rangle = \int d\mathbf{c}\chi_H^{-1}(c)f^*(\mathbf{c})g(\mathbf{c}), \tag{34}$$

with f^* the complex conjugate of f.

3 The Boltzmann-Langevin equation

The starting point for the study of the fluctuations in this work will be the Boltzmann-Langevin equation. In this section we derive the corresponding equation and we determine the properties of the noise in order to obtain consistency with the equations of the correlation functions presented in the previous section.

As defined in Eq. (5), the one-particle distribution function, f_1, is the ensemble average of the phase function F_1, and its dynamics is given by the Boltzmann-Fokker-Planck equation, Eq. (7). The problem is now to find an evolution equation for the fluctuating quantity

$$\delta F(\mathbf{l},\mathbf{c},s) = \frac{v_H^d}{n_H}[F_1(x,t) - f_H(\mathbf{v})]. \tag{35}$$

As for the velocity of a Brownian particle [30], we expect that the difference between the equation for the fluctuating quantity δF, and its average, $\langle\delta F\rangle = \delta\chi$, is a random force term, R [31]. Then, the fluctuations δF around χ_H are described by a Boltzmann equation linearized around the solution χ_H with a random force, R, added

$$\frac{\partial}{\partial s}\delta F(\mathbf{l},\mathbf{c},s) = \left[\Lambda(\mathbf{c}) - \mathbf{c}\cdot\frac{\partial}{\partial \mathbf{l}}\right]\delta F(\mathbf{l},\mathbf{c},s) + R(\mathbf{l},\mathbf{c},s). \tag{36}$$

Taking averages in equation (36), we obtain the linearized Boltzmann-Fokker-Planck equation, Eq. (29), if and only if

$$\langle R(\mathbf{l},\mathbf{c},s)\rangle = 0. \tag{37}$$

Equation (36) is the Boltzmann-Langevin equation. As in the free cooling case [15], we assume that the noise term, $R(\mathbf{l},\mathbf{c},s)$, is Markovian

$$\langle R(\mathbf{l}_1,\mathbf{c}_1,s_1)R(\mathbf{l}_2,\mathbf{c}_2,s_2)\rangle_H = H(\mathbf{l}_1,\mathbf{l}_2,\mathbf{c}_1,\mathbf{c}_2)\delta(s_1 - s_2), \tag{38}$$

where $\langle\dots\rangle_H$ means average in the stationary homogeneous state. In order to evaluate the function $H(\mathbf{l}_1,\mathbf{l}_2,\mathbf{c}_1,\mathbf{c}_2)$ explicitely, we will calculate $\langle\delta F(\mathbf{l}_1,\mathbf{c}_1,s)\delta F(\mathbf{l}_2,\mathbf{c}_2,s)\rangle_H$ with the Boltzmann-Langevin equation and then, we will impose compatibility with the equation of the correlation function of the previous section. So, let us first write this function as a functional of the distribution and the correlation functions. Using the definitions of the microscopic densities, Eqs. (3), (4), the distribution and correlation functions, Eqs. (5), (11), (22), and the dimensionless distributions, Eqs. (15), (19), we have

$$\begin{aligned}\langle\delta F(x_1,t)\delta F(x_2,t)\rangle_H &= \frac{v_H^{2d}}{n_H^2}\langle[F_1(x_1,t)-f_H(\mathbf{v}_1)][F_1(x_2,t)-f_H(\mathbf{v}_2)]\rangle_H \\ &= \frac{v_H^{2d}}{n_H^2}[\langle F_1(x_1,t)F_1(x_2,t)\rangle_H - f_H(\mathbf{v}_1)f_H(\mathbf{v}_2)] \\ &= \frac{v_H^{2d}}{n_H^2}[f_{2,H}(x_1,x_2,t)+f_H(\mathbf{v}_1)\delta(x_1-x_2)-f_H(\mathbf{v}_1)f_H(\mathbf{v}_2)] \\ &= \frac{v_H^{2d}}{n_H^2}[g_{2,H}(x_1,x_2)+f_H(\mathbf{v}_1)\delta(x_1-x_2)] \\ &= \frac{1}{n_H\ell^d}[\tilde{g}_H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2)+\delta(\mathbf{l}_{12})\delta(\mathbf{c}_{12})\chi_H(c_1)] \\ &= \frac{1}{n_H\ell^d}h_H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2). \end{aligned} \tag{39}$$

Now, let us solve Eq. (36) as a functional of the noise. In order to do that it is convenient to define the linear operator

$$\Lambda(\mathbf{l}_i,\mathbf{c}_i) \equiv \Lambda(\mathbf{c}_i) - \mathbf{c}_i\cdot\frac{\partial}{\partial\mathbf{l}_i}, \tag{40}$$

in terms of which the solution for $\delta F(\mathbf{l},\mathbf{c},s)$ is

$$\begin{aligned}\delta F(\mathbf{l},\mathbf{c},s) &= e^{\Lambda(\mathbf{l},\mathbf{c})s}\delta F(\mathbf{l},\mathbf{c},0) + \int_0^s \mathrm{d}s' e^{\Lambda(\mathbf{l},\mathbf{c})(s-s')}R(\mathbf{l},\mathbf{c},s') \\ &\overset{s\gg 1}{\rightarrow} \int_0^s \mathrm{d}s' e^{\Lambda(\mathbf{l},\mathbf{c})(s-s')}R(\mathbf{l},\mathbf{c},s'), \end{aligned} \tag{41}$$

where we have assumed that the term stemming from the initial conditions vanishes in the long time limit. This is equivalent to assuming that the spectrum of the linearized Boltzmann-Fokker-Planck operator is such that any perturbation without component in the subspace associated to the null eigenvalue decays. With equation (41), we can evaluate

$$\begin{aligned}&\langle\delta F(\mathbf{l}_1,\mathbf{c}_1,s)\delta F(\mathbf{l}_2,\mathbf{c}_2,s)\rangle_H \\ &= \int_0^s \mathrm{d}s'\int_0^s \mathrm{d}s'' e^{\Lambda(\mathbf{l}_1,\mathbf{c}_1)(s-s')+\Lambda(\mathbf{l}_2,\mathbf{c}_2)(s-s'')}\langle R(\mathbf{l}_1,\mathbf{c}_1,s')R(\mathbf{l}_2,\mathbf{c}_2,s'')\rangle_H \\ &= \int_0^s \mathrm{d}s'\int_0^s \mathrm{d}s'' e^{\Lambda(\mathbf{l}_1,\mathbf{c}_1)(s-s')+\Lambda(\mathbf{l}_2,\mathbf{c}_2)(s-s'')}H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2)\delta(s'-s'') \\ &= \int_0^s \mathrm{d}s' e^{[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)](s-s')}H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2) \\ &= -[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)]^{-1}\left[e^{[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)](s-s')}\right]_{s'=0}^{s'=s}H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2) \\ &\overset{s\gg 1}{\rightarrow} -[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)]^{-1}H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2), \end{aligned} \tag{42}$$

where we have assumed that the term $e^{[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)]s}H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2)\to 0$ in the long time limit. After identifying the function, we will see that this is, in fact, the case, because $H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2)$ does not have components in the subspace associated to the null eigenvalue. Equivalently we have

$$[\Lambda(\mathbf{l}_1,\mathbf{c}_1)+\Lambda(\mathbf{l}_2,\mathbf{c}_2)]h_H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2) = -n_H\ell^d H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2). \tag{43}$$

Finally, comparing equations (43) and (23) we conclude that

$$H(\mathbf{l}_{12},\mathbf{c}_1,\mathbf{c}_2) = \frac{1}{n_H\ell^d}\delta(\mathbf{l}_{12})\Gamma(\mathbf{c}_1,\mathbf{c}_2) - \frac{\tilde{\xi}^2}{N}\frac{\partial}{\partial\mathbf{c}_1}\cdot\frac{\partial}{\partial\mathbf{c}_2}\chi_H(c_1)\chi_H(c_2), \tag{44}$$

with $\Gamma(\mathbf{c}_1, \mathbf{c}_2)$ given in Eq. (24). With this expression of H we can see that it does not have components in the subspace associated to the null eigenvalue. Taking into account that $\bar{\xi}_1$ and $\bar{\boldsymbol{\xi}}_2$ are left eigenfunctions of Λ associated to the null eigenvalue and that [32]

$$\int d\mathbf{c}_1 \int d\mathbf{c}_2 \bar{T}(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) = 0,$$
$$\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{2i} \bar{T}(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) = \tilde{\xi}^2, \tag{45}$$

we can see that

$$\int d\mathbf{l}_{12} \int d\mathbf{c}_1 \int d\mathbf{c}_2 H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2) = 0,$$
$$\int d\mathbf{l}_{12} \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{2i} H(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2) = 0. \tag{46}$$

In the remaining of this section, we shall write the Boltzmann-Langevin equation together with noise properties in Fourier space. This will prove useful for the subsequent analysis. Let us introduce the Fourier component of a function of the position variable as

$$\tilde{f}(\mathbf{k}) = \int d\mathbf{l} e^{-i\mathbf{k}\cdot\mathbf{l}} f(\mathbf{l}), \qquad f(\mathbf{l}) = \frac{1}{\tilde{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{l}} \tilde{f}(\mathbf{k}), \tag{47}$$

where $\tilde{V} = \frac{V}{\ell^d}$ is the volume in units of the mean free path. The equation for $\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s)$ is then

$$\left[\frac{\partial}{\partial s} - \Lambda(\mathbf{k}, \mathbf{c})\right] \delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) = \tilde{R}(\mathbf{k}, \mathbf{c}, s), \tag{48}$$

where we have introduced the operator

$$\Lambda(\mathbf{k}, \mathbf{c}) = \Lambda(\mathbf{c}) - i\mathbf{k}\cdot\mathbf{c}. \tag{49}$$

The Fourier transform of the noise, $\tilde{R}(\mathbf{k}, \mathbf{c}, s)$, obeys

$$\langle \tilde{R}(\mathbf{k}, \mathbf{c}, s)\rangle_H = 0, \tag{50}$$

and

$$\langle \tilde{R}(\mathbf{k}_1, \mathbf{c}_1, s_1)\tilde{R}(\mathbf{k}_2, \mathbf{c}_2, s_2)\rangle_H = H(\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_1, \mathbf{c}_2)\delta(s_1 - s_2), \tag{51}$$

where

$$H(\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_1, \mathbf{c}_2) = \frac{\tilde{V}^2}{N}\left[\Gamma(\mathbf{c}_1, \mathbf{c}_2)\delta(\mathbf{k}_1 + \mathbf{k}_2) - \tilde{\xi}^2 \frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{\partial}{\partial \mathbf{c}_2} \chi_H(c_1)\chi_H(c_2)\delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\right], \tag{52}$$

which completes the calculation of the noise variance within the Langevin description, that will be used to quantify the transverse velocity fluctuations.

4 The fluctuating transverse velocity

The objective in this section is to derive a fluctuating equation for the transverse velocity field, $\mathbf{w}_\perp(\mathbf{k}, s)$. The reason to consider this field is that its equation is decoupled from the equations for the other hydrodynamic fields [20], and it can be derived exactly in the hydrodynamic limit with the knowledge we have about the spectrum of the linearized Boltzmann-Fokker-Planck operator.

Mathematically, the transverse velocity is defined in the following way: Let us consider the d-dimensional vector $\mathbf{k}$ which belongs to the space $\Re^d$. This space can be expanded in the base

$\left\{\hat{\mathbf{k}}\right\} \bigcup \left\{\hat{\mathbf{k}}_{\perp}^{i}\right\}_{i=1}^{d-1}$ where $\hat{\mathbf{k}}$ is a unitary vector parallel to $\mathbf{k}$ and $\left\{\hat{\mathbf{k}}_{\perp}^{i}\right\}_{i=1}^{d-1}$ are $(d-1)$ unitary vectors orthogonal to $\mathbf{k}$. The transverse velocity is defined as

$$w_{\perp i}(\mathbf{k}, s) = \int d\mathbf{c}(\mathbf{c} \cdot \hat{\mathbf{k}}_{\perp}^{i})\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s), \qquad i = 1, \ldots, d-1. \tag{53}$$

For the subsequent analysis, is convenient to introduce the following projectors

$$P^{(i)} f(\mathbf{c}) \equiv \langle \bar{\xi}_{2\perp i}(\mathbf{c})|f(\mathbf{c})\rangle \xi_{2\perp i}(\mathbf{c}), \tag{54}$$

$$Q^{(i)} f(\mathbf{c}) \equiv (1 - P^{(i)}) f(\mathbf{c}). \tag{55}$$

Then, if we apply $P^{(i)}$ to $\delta\tilde{F}$ we obtain

$$P^{(i)}\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) = w_{\perp i}(\mathbf{k}, s)\xi_{2\perp i}(\mathbf{c}), \tag{56}$$

and the transverse velocity is the component of $\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s)$ in the subspace generated by $\xi_{2\perp i}(\mathbf{c})$.

In order to obtain an equation for $\mathbf{w}_{\perp}$, we apply the projectors P and Q (for simplicity in the notation we skip the super-index) to the Langevin equation (48)

$$\left[\frac{\partial}{\partial s} - P\Lambda(\mathbf{k}, \mathbf{c})\right] P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) = P\tilde{R}(\mathbf{k}, \mathbf{c}, s) - Pi(\mathbf{k} \cdot \mathbf{c})Q\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s), \tag{57}$$

$$\left[\frac{\partial}{\partial s} - Q\Lambda(\mathbf{k}, \mathbf{c})\right] Q\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) = Q\tilde{R}(\mathbf{k}, \mathbf{c}, s) - Qi(\mathbf{k} \cdot \mathbf{c})P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s). \tag{58}$$

Now, let us solve equation (58) formally

$$\begin{aligned} Q\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) &= e^{Q\Lambda(\mathbf{k},\mathbf{c})s} Q\delta\tilde{F}(\mathbf{k}, \mathbf{c}, 0) \\ &\quad + \int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')}[Q\tilde{R}(\mathbf{k}, \mathbf{c}, s') - Qi(\mathbf{k} \cdot \mathbf{c})P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s')]. \end{aligned} \tag{59}$$

In the time regime in which the system has forgotten the initial condition, i.e. when we can consider that $e^{Q\Lambda(\mathbf{k},\mathbf{c})s}Q\delta\tilde{F}(\mathbf{k}, \mathbf{c}, 0) \to 0$, by substituting Eq. (59) in Eq. (57), we obtain a closed equation for $P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s)$

$$\begin{aligned} &\left[\frac{\partial}{\partial s} - P\Lambda(\mathbf{k}, \mathbf{c})\right] P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s) + P(\mathbf{k} \cdot \mathbf{c}) \int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')} Q(\mathbf{k} \cdot \mathbf{c})P\delta\tilde{F}(\mathbf{k}, \mathbf{c}, s') \\ &= P\tilde{R}(\mathbf{k}, \mathbf{c}, s) - Pi(\mathbf{k} \cdot \mathbf{c}) \int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')} Q\tilde{R}(\mathbf{k}, \mathbf{c}, s'). \end{aligned} \tag{60}$$

In Appendix A it is shown that, in the hydrodynamic limit, i.e. to second order in k and in the long time limit, Eq. (60) reduces to the following equation for the transverse velocity

$$\left[\frac{\partial}{\partial s} + \tilde{\eta}k^2\right] w_{\perp}(\mathbf{k}, s) = R_w(\mathbf{k}, s). \tag{61}$$

The coefficient $\tilde{\eta}$ is the shear viscosity given by

$$\tilde{\eta} = \int_0^{\infty} \mathrm{d}s \int \mathrm{d}\mathbf{c} c_x c_y e^{\Lambda(\mathbf{c})s} c_x \xi_{2,y}(\mathbf{c}) = -\int \mathrm{d}\mathbf{c} c_x c_y \Lambda(\mathbf{c})^{-1} c_x \xi_{2,y}(\mathbf{c}), \tag{62}$$

which agrees with the expression obtained in [24] by the Chapman-Enskog method, and $R_w(\mathbf{k}, s)$ is a noise term which can be decomposed as

$$R_w(\mathbf{k}, s) = S(\mathbf{k}, s) + \Pi(\mathbf{k}, s). \tag{63}$$

The term $S(\mathbf{k},s)$ comes from the thermostat (which does not conserve momentum locally), and the second term, $\Pi(\mathbf{k},s)$, is the fluctuating part of the pressure tensor. Their microscopic expressions in terms of the noise of the Boltzmann-Langevin equation are

$$S(\mathbf{k},s) = \int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}}_\perp \cdot \mathbf{c})\tilde{R}(\mathbf{k},\mathbf{c},s), \tag{64}$$

and

$$\Pi(\mathbf{k},s) = -ik\int_0^s \mathrm{d}s' \int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}} \cdot \mathbf{c})(\hat{\mathbf{k}}_\perp \cdot \mathbf{c})e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')}Q\tilde{R}(\mathbf{k},\mathbf{c},s'). \tag{65}$$

Of course, due to the fact that $\langle \tilde{R}(\mathbf{k},\mathbf{c},s)\rangle_H = 0$, the mean value of the noise vanishes

$$\langle R_w(\mathbf{k},s)\rangle_H = 0. \tag{66}$$

The autocorrelation function of the noise is evaluated in Appendix B. Due to symmetry considerations, there are only correlations between the $\mathbf{k}$ and $-\mathbf{k}$ components, which yields

$$\begin{aligned}\langle R_w(\mathbf{k},s_1)R_w(-\mathbf{k},s_2)\rangle_H = \frac{\tilde{V}^2}{N}\left[\tilde{\xi}^2\delta(s_1-s_2) + k^2C_{xy}(s_2-s_1)\right]\\ + \langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H, \qquad s_1 < s_2.\end{aligned} \tag{67}$$

The first term is the expected zeroth order term which comes from the heating. The Dirac delta function is an exact consequence of the fact that the external noise (the heating) is white. The function $C_{xy}(s_2-s_1)$ reads

$$C_{xy}(s_2-s_1) = \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x}c_{1y}c_{2x}c_{2y}e^{\Lambda(\mathbf{c}_2)(s_2-s_1)}\phi_H(\mathbf{c}_1,\mathbf{c}_2). \tag{68}$$

Here, we have introduced the function $\phi_H(\mathbf{c}_1,\mathbf{c}_2)$ as the space integral of the correlation function $h_H(\mathbf{l},\mathbf{c}_1,\mathbf{c}_2)$

$$\phi_H(\mathbf{c}_1,\mathbf{c}_2) = \int \mathrm{d}\mathbf{l}h_H(\mathbf{l},\mathbf{c}_1,\mathbf{c}_2) = \chi_H(c_1)\delta(\mathbf{c}_{12}) + \chi_2(\mathbf{c}_1,\mathbf{c}_2), \tag{69}$$

where $\chi_2(\mathbf{c}_1,\mathbf{c}_2) = \int d\mathbf{l}\tilde{g}_H(\mathbf{l},\mathbf{c}_1,\mathbf{c}_2)$ and $\tilde{g}_H$ is the dimensionless correlation function defined in (19). The equation for ϕ_H can easily be obtained by integration over space variable in Eq. (23), which gives

$$[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2)]\phi_H(\mathbf{c}_1,\mathbf{c}_2) = -\Gamma(\mathbf{c}_1,\mathbf{c}_2) + \tilde{\xi}^2\frac{\partial}{\partial\mathbf{c}_1}\cdot\frac{\partial}{\partial\mathbf{c}_2}\chi_H(c_1)\chi_H(c_2). \tag{70}$$

As discussed in Appendix B, C_{xy} can be physically interpreted as the autocorrelation function of the global quantity $\sum_{i=1}^N V_x(t)V_y(t)$. In the elastic case, this correlation function is related to the shear viscosity but it is not the case for granular systems [34]. The formula for the correlation $\langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H$ is given in Appendix B, and it does not seem to admit a direct physical interpretation. As the two correlation functions, $C_{xy}(s)$ and $\langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H$, decay with the kinetic modes and, in the $k \to 0$ limit, the fluctuating velocity is expected to be frozen (its time evolution is given by the null eigenvalue, Eq. (61)), we can consider that they are proportional to a Dirac delta function in time, and we have

$$\begin{aligned}&\frac{\tilde{V}^2}{N}k^2C_{xy}(s_2-s_1) + \langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H\\ &\to 2\left[\frac{\tilde{V}^2}{N}k^2\int_0^\infty \mathrm{d}sC_{xy}(s) + \int_0^\infty \mathrm{d}s\langle S(\mathbf{k},0)\Pi(-\mathbf{k},s)\rangle_H\right]\delta(s_1-s_2)\,.\end{aligned} \tag{71}$$

Note that this is in contrast with the free cooling case where the noise can not be considered to be white [15]. In this case, the equation for the transverse velocity (rescaled with the thermal velocity) contains a term of order zero in k which is proportional to the cooling rate. Then, even in the $k \to 0$ limit, the velocity is not frozen on the kinetic scale. In Appendix C the second integral of equation (71) is evaluated obtaining

$$\int_0^\infty \mathrm{d}s \langle S(\mathbf{k},0)\Pi(-\mathbf{k},s)\rangle_H \to \frac{\tilde{V}^2}{N} k^2 \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} c_{2y} \phi_H(\mathbf{c}_1,\mathbf{c}_2), \tag{72}$$

which is valid in the hydrodynamic limit. If we substitute $\phi_H(\mathbf{c}_1,\mathbf{c}_2)$ by its expression in terms of the one and two-particle distribution function, Eq. (69), we obtain (see Appendix D) that the one-particle contribution vanishes and the correlation function can be written in terms of the two-particle velocity correlation function, $\chi_2(\mathbf{c}_1,\mathbf{c}_2)$

$$\begin{aligned} &\langle R_w(\mathbf{k},s_1) R_w(-\mathbf{k},s_2)\rangle_H \\ &= \frac{\tilde{V}^2}{N}\delta(s_1-s_2)\left\{\tilde{\xi}^2 + 2k^2\left[-\int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x}c_{1y}c_{2x}c_{2y}\Lambda(\mathbf{c}_2)^{-1}\chi_2(\mathbf{c}_1,\mathbf{c}_2)\right.\right. \\ &\quad \left.\left. + \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x}c_{2x}c_{2y}\Lambda(\mathbf{c}_2)^{-1}c_{2y}\chi_2(\mathbf{c}_1,\mathbf{c}_2)\right]\right\}. \end{aligned} \tag{73}$$

As $R_w(\mathbf{k},s_1)$ is Gaussian, the noise is completely characterized by Equations (66) and (73). As it can be seen, the k^2 term has no relation, a priori, with the shear viscosity, Eq. (62), i.e. there is a priori no fluctuation-dissipation relation as that assumed in [20]. *However*, we now show that, under additional hypothesis (that in principle are not restricted to the elastic limit), the aforementioned term reduces to the shear viscosity. Let us assume that the most important contribution of the two particle velocity correlation function, $\chi_2(\mathbf{c}_1,\mathbf{c}_2)$, is the hydrodynamic part, i.e. we assume

$$\chi_2(\mathbf{c}_1,\mathbf{c}_2) \simeq \sum_{\beta=1}^{d+2}\sum_{\beta'=1}^{d+2} a_{\beta,\beta'}\xi_\beta(\mathbf{c}_1)\xi_{\beta'}(\mathbf{c}_2). \tag{74}$$

This assumption was already made in [32] where the coefficients $a_{\beta,\beta'}$ were evaluated to calculate the total energy fluctuations. We emphasize that it led to an excellent agreement between analytical predictions and numerical data (Monte Carlo) for energy fluctuations, for all the values of the inelasticity [32]. In this approximation, the first integral in (73) vanishes because $c_x c_y \chi_H(c)$ is orthogonal to the hydrodynamic modes. The second term is

$$\begin{aligned} &\int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x}c_{2x}c_{2y}\Lambda(\mathbf{c}_2)^{-1}c_{2y}\sum_{\beta=1}^{d+2}\sum_{\beta'=1}^{d+2} a_{\beta,\beta'}\xi_\beta(\mathbf{c}_1)\xi_{\beta'}(\mathbf{c}_2) \\ &= a_{2x,2x}\int \mathrm{d}\mathbf{c}_1 c_{1x}\xi_{2x}(\mathbf{c}_1)\int \mathrm{d}\mathbf{c}_2 c_{2x}c_{2y}\Lambda(\mathbf{c}_2)^{-1}c_{2y}\xi_{2x}(\mathbf{c}_2), \end{aligned} \tag{75}$$

where, for symmetry considerations, the only term that remains is the one associated to $\beta = \beta' = 2$. If we use now that $a_{2i2i} = -1/2$ (see the reference [32]), and the formula for the shear viscosity, Eq. (62), we finally have

$$\langle R_w(\mathbf{k},s_1)R_w(-\mathbf{k},s_2)\rangle = \frac{\tilde{V}^2}{N}\delta(s_1-s_2)(\tilde{\xi}^2 + \tilde{\eta}k^2). \tag{76}$$

Then, in the hydrodynamic limit and assuming that the two-particle velocity correlation function, $\chi_2(\mathbf{c}_1,\mathbf{c}_2)$, has only components in the hydrodynamic subspace, the correlation function of the noise reduces to the one introduced phenomenologically in [20], not only in the elastic limit, but for arbitrary inelasticity. Although we have no direct proof of the accuracy of approximation (74), we note that it is backed up by numerical data, see e.g. [32].

5 Conclusions and outlook

The primary objective of this work was to derive a Boltzmann-Langevin description for a heated granular gas, in the spirit of the work of Bixon and Zwanzig [31], who considered conservative systems. The system considered here is on the other hand dissipative, and is heated by a stochastic force changing the velocity of the particles between collisions. The loss of energy in collisions is compensated by the energy given to the particles by the thermostat and a stationary state is thereby reached. Our study is restricted to this stationary state. The Boltzmann-Langevin equation has been derived and the properties of the noise appearing in this equation were identified by imposing consistency with the equation for the correlation function derived in [32].

The Boltzmann-Langevin equation is the starting point for the derivation of fluctuating hydrodynamic equations. This can be done by projecting the Boltzmann-Langevin equation into the hydrodynamic subspace. As our knowledge of the spectrum of the linearized Boltzmann-Fokker-Planck operator is quite limited, we have focused on the equation for the transverse velocity field, that is decoupled from the rest of the fluctuating hydrodynamic equations. This specific case was studied in [14,15] for the free cooling state, where it was shown that the relevant Langevin noise is not white and that there is no fluctuation-dissipation relation. In other words, the amplitude of the noise is not related to the shear viscosity. On the other hand, in the stochastically heated system, the behavior that we have reported is different. First, the noise of the transverse velocity contains two parts: one coming from the thermostat (which does not conserve momentum locally) together with the more standard fluctuating pressure tensor. The correlation function of the noise can then be written as a sum of direct and cross terms made up from the previous two contributions. Moreover, in contrast to the free cooling scenario, the noise can be considered as white [36], as the dynamics of the velocity is as slow as desired in the hydrodynamic (low k) limit. As in the free cooling case, in the hydrodynamic limit, the amplitude of the noise is *a priori* not related to the shear viscosity (such a relation, of fluctuation-dissipation type, had been assumed in the approach of Ref [20]). However, considering that the two-particle velocity correlation function has only hydrodynamic modes – which seems a reasonable assumption – somehow restores fluctuation-dissipation and we obtain the expression assumed in [20], with the actual inelastic shear viscosity. In principle, this rather surprising result – reminiscent of those reported in Refs [37] – is not limited to small inelasticity.

For future work, remains the extension of the theory to the other hydrodynamic equations (beyond the transverse velocity), together with a generalization of the scheme presented here to a more general class of thermostated systems, such as the ones with multiplicative noise [35].

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Appendix

A Derivation of the transverse velocity field equation

In this Appendix we derive the equation for the transverse velocity field, $w_\perp(\mathbf{k}, s)$, in the hydrodynamic limit. The starting point is equation (60)

$$\left[\frac{\partial}{\partial s} - P\Lambda(\mathbf{k},\mathbf{c})\right] P\delta\tilde{F}(\mathbf{k},\mathbf{c},s) + P(\mathbf{k}\cdot\mathbf{c})\int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')} Q(\mathbf{k}\cdot\mathbf{c}) P\delta\tilde{F}(\mathbf{k},\mathbf{c},s')$$
$$= P\tilde{R}(\mathbf{k},\mathbf{c},s) - Pi(\mathbf{k}\cdot\mathbf{c})\int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')} Q\tilde{R}(\mathbf{k},\mathbf{c},s'). \quad (77)$$

Let us first consider the term $P\Lambda(\mathbf{k},\mathbf{c})P\delta\tilde{F}(\mathbf{k},\mathbf{c},s)$. As $\Lambda(\mathbf{c})\xi_{2\perp}(\mathbf{c}) = 0$ and $\int d\mathbf{c}(\hat{\mathbf{k}}_\perp \cdot \mathbf{c})(\hat{\mathbf{k}} \cdot \mathbf{c})\xi_{2\perp}(\mathbf{c}) = 0$, we easily have

$$P\Lambda(\mathbf{k},\mathbf{c})P\delta\tilde{F}(\mathbf{k},\mathbf{c},s) = 0. \quad (78)$$

Let us evaluate the last term of the left hand side of Eq. (77). To second order in k, we have

$$
\begin{aligned}
&P(\mathbf{k}\cdot\mathbf{c})\int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')}Q(\mathbf{k}\cdot\mathbf{c})P\delta\tilde{F}(\mathbf{k},\mathbf{c},s') \\
&\simeq k^2\xi_{2\perp}(\mathbf{c})\int_0^s \mathrm{d}s' w_\perp(\mathbf{k},s')\int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}}\cdot\mathbf{c})(\hat{\mathbf{k}}_\perp\cdot\mathbf{c})e^{\Lambda(\mathbf{c})(s-s')}\hat{\mathbf{k}}\cdot\mathbf{c}\xi_{2\perp}(\mathbf{c}) \\
&= k^2\xi_{2\perp}(\mathbf{c})\int_0^s \mathrm{d}s' w_\perp(\mathbf{k},s')G_{xy}(s-s') \qquad (79)
\end{aligned}
$$

where we have introduced

$$
G_{xy}(s)\equiv\int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}}_\perp\cdot\mathbf{c})(\hat{\mathbf{k}}\cdot\mathbf{c})e^{\Lambda(\mathbf{c})s}(\hat{\mathbf{k}}\cdot\mathbf{c})\xi_{2\perp}(\mathbf{c})=\int \mathrm{d}\mathbf{c}c_xc_ye^{\Lambda(\mathbf{c})s}c_x\xi_{2y}(\mathbf{c}), \qquad (80)
$$

and use has been made of the fact that the operator $\Lambda(\mathbf{c})$ is isotropic. In the hydrodynamic limit, the velocity evolves in a scale much slower that the scale in which the function $G_{xy}(s)$ decays. We then have

$$
\int_0^s \mathrm{d}s' w_\perp(\mathbf{k},s')G_{xy}(s-s')\to\tilde{\eta}w_\perp(\mathbf{k},s), \qquad (81)
$$

where $\tilde{\eta}$ is the dimensionless shear viscosity

$$
\tilde{\eta}=\int_0^\infty \mathrm{d}sG_{xy}(s). \qquad (82)
$$

The noise terms are the last two terms of Eq. (77)

$$
P\tilde{R}(\mathbf{k},\mathbf{c},s)=\xi_{2\perp}(\mathbf{c})\int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}}_\perp\cdot\mathbf{c})\tilde{R}(\mathbf{k},\mathbf{c},s)=\xi_{2\perp}(\mathbf{c})S(\mathbf{k},s), \qquad (83)
$$

and

$$
\begin{aligned}
&P(i\mathbf{k}\cdot\mathbf{c})\int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')}Q\tilde{R}(\mathbf{k},\mathbf{c},s') \\
&=\xi_{2\perp}(\mathbf{c})ik\int \mathrm{d}\mathbf{c}(\hat{\mathbf{k}}_\perp\cdot\mathbf{c})(\hat{\mathbf{k}}\cdot\mathbf{c})\int_0^s \mathrm{d}s' e^{Q\Lambda(\mathbf{k},\mathbf{c})(s-s')}Q\tilde{R}(\mathbf{k},\mathbf{c},s') \\
&=-\xi_{2\perp}(\mathbf{c})\Pi(\mathbf{k},s), \qquad (84)
\end{aligned}
$$

where we have used the definitions of $S(\mathbf{k},s)$ and the fluctuating pressure tensor, $\Pi(\mathbf{k},s)$, Eqs. (64) and (65). Finally, by multiplying Eq. (77) by $\hat{\mathbf{k}}_\perp\cdot\mathbf{c}$ and further integrating over velocities, we obtain the equation of the transverse velocity of the main text.

B Autocorrelation function of $R_w(\mathbf{k},s)$

In this Appendix we evaluate the correlation function of the noise of the transverse velocity field, $R_w(\mathbf{k},s)$. We consider $\mathbf{k}\neq\mathbf{0}$, $s_1<s_2$ with s_1 large. It is convenient to introduce the following notation for the transverse and parallel components of the vector $\mathbf{c}$

$$
\hat{\mathbf{k}}_\perp\cdot\mathbf{c}=c_\perp,\quad \hat{\mathbf{k}}\cdot\mathbf{c}=c_\parallel. \qquad (85)
$$

The autocorrelation function of $R_w(\mathbf{k},s)$ reads, in terms of $S(\mathbf{k},s)$ and $\Pi(\mathbf{k},s)$,

$$
\begin{aligned}
\langle R_w(\mathbf{k},s_1)R_w(-\mathbf{k},s_2)\rangle_H &= \langle(S(\mathbf{k},s_1)+\Pi(\mathbf{k},s_1))(S(-\mathbf{k},s_2)+\Pi(-\mathbf{k},s_2))\rangle_H \\
&= \langle S(\mathbf{k},s_1)S(-\mathbf{k},s_2)\rangle_H+\langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H \\
&\quad+\langle\Pi(\mathbf{k},s_1)S(-\mathbf{k},s_2)\rangle_H+\langle\Pi(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H. \qquad (86)
\end{aligned}
$$

We now calculate each correlation function taking into account the microscopic expressions of $S(\mathbf{k},s)$ and $\Pi(\mathbf{k},s)$, Eqs. (64) and (65), and the correlation function of the noise of the Boltzmann-Langevin equation, Eqs. (51) and (52).

The first term is

$$\begin{aligned}\langle S(\mathbf{k},s_1)S(-\mathbf{k},s_2)\rangle_H &= \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1\perp}c_{2\perp}\langle\tilde{R}(\mathbf{k},\mathbf{c}_1,s_1)\tilde{R}(-\mathbf{k},\mathbf{c}_2,s_2)\rangle_H \\ &= \frac{\tilde{V}^2}{N}\delta(s_1-s_2)\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\perp}c_{2\perp}\Gamma(\mathbf{c}_1,\mathbf{c}_2) \\ &= \frac{\tilde{V}^2}{N}\delta(s_1-s_2)\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\perp}c_{2\perp}\bar{T}(\mathbf{c}_1,\mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) \\ &= \tilde{\xi}^2\frac{\tilde{V}^2}{N}\delta(s_1-s_2). \end{aligned} \tag{87}$$

where we have used the relation $\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\perp}c_{2\perp}\bar{T}(\mathbf{c}_1,\mathbf{c}_2)\chi_H(c_1)\chi_H(c_2)=\tilde{\xi}^2$, that is proved in [32].

The second correlation function is

$$\begin{aligned}&\langle S(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H \\ &\quad= \int d\mathbf{c}_1 c_{1\perp} ik\int_0^{s_2} ds\int d\mathbf{c}_2 c_{2\parallel}c_{2\perp}e^{Q_2\Lambda(-\mathbf{k},\mathbf{c}_2)(s_2-s)}\langle\tilde{R}(\mathbf{k},\mathbf{c}_1,s_1)Q_2\tilde{R}(-\mathbf{k},\mathbf{c}_2,s)\rangle_H \\ &\quad= ik\frac{\tilde{V}^2}{N}\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\perp}c_{2\parallel}c_{2\perp}e^{Q_2\Lambda(-\mathbf{k},\mathbf{c}_2)(s_2-s_1)}Q_2\Gamma(\mathbf{c}_1,\mathbf{c}_2). \end{aligned} \tag{88}$$

Here, we have changed the sign of $\Pi(-\mathbf{k},s)$ because we are dealing with the $-\mathbf{k}$ component and then $\hat{\mathbf{k}}\to-\hat{\mathbf{k}}$ (we do not change $\hat{\mathbf{k}}_\perp\to-\hat{\mathbf{k}}_\perp$, because this vector comes from the projector P and it is fixed).

The third term vanishes

$$\begin{aligned}\langle\Pi(\mathbf{k},s_1)S(-\mathbf{k},s_2)\rangle_H &= -ik\int_o^{s_1} ds'\int d\mathbf{c}_1 c_{1\parallel}c_{1\perp}\int d\mathbf{c}_2 c_{2\perp} \\ &\quad\times e^{Q_1\Lambda(\mathbf{k},\mathbf{c}_1)(s-s')}Q_1\langle\tilde{R}(\mathbf{k},\mathbf{c}_1,s')\tilde{R}(-\mathbf{k},\mathbf{c}_2,s_2)\rangle_H = 0, \end{aligned} \tag{89}$$

because $\langle\tilde{R}(s')\tilde{R}(s_2)\rangle_H=0$ for $s'\in(0,s_1)$ with $s_1<s_2$.

Finally, we evaluate the last term to second order in k

$$\begin{aligned}&\langle\Pi(\mathbf{k},s_1)\Pi(-\mathbf{k},s_2)\rangle_H \\ &\quad\simeq k^2\int_0^{s_1}ds_1'\int_0^{s_2}ds_2'\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\parallel}c_{1\perp}c_{2\parallel}c_{2\perp}e^{\Lambda(\mathbf{c}_1)(s_1-s_1')+\Lambda(\mathbf{c}_2)(s_2-s_2')} \\ &\qquad\times\langle\tilde{R}(\mathbf{k},\mathbf{c}_1,s_1')\tilde{R}(-\mathbf{k},\mathbf{c}_2,s_2')\rangle_H \\ &\quad= \frac{\tilde{V}^2}{N}k^2\int_0^{s_1}ds_1'\int_0^{s_2}ds_2'\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\parallel}c_{1\perp}c_{2\parallel}c_{2\perp}e^{\Lambda(\mathbf{c}_1)(s_1-s_1')+\Lambda(\mathbf{c}_2)(s_2-s_2')}\Gamma(\mathbf{c}_1,\mathbf{c}_2)\delta(s_1'-s_2') \\ &\quad= \frac{\tilde{V}^2}{N}k^2\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\parallel}c_{1\perp}c_{2\parallel}c_{2\perp}\int_0^{s_1} ds\, e^{\Lambda(\mathbf{c}_1)(s_1-s)+\Lambda(\mathbf{c}_2)(s_2-s)}\Gamma(\mathbf{c}_1,\mathbf{c}_2) \\ &\quad= \frac{\tilde{V}^2}{N}k^2\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\parallel}c_{1\perp}c_{2\parallel}c_{2\perp}e^{\Lambda(\mathbf{c}_1)s_1+\Lambda(\mathbf{c}_2)s_2}\int_0^{s_1} ds\, e^{-s[\Lambda(\mathbf{c}_1)+\Lambda(\mathbf{c}_2)]}\Gamma(\mathbf{c}_1,\mathbf{c}_2) \\ &\simeq \frac{\tilde{V}^2}{N}k^2\int d\mathbf{c}_1\int d\mathbf{c}_2 c_{1\parallel}c_{1\perp}c_{2\parallel}c_{2\perp}e^{\Lambda(\mathbf{c}_2)(s_2-s_1)} \\ &\qquad\times[\Lambda(\mathbf{c}_1)+\Lambda(\mathbf{c}_2)]^{-1}\left[-\Gamma(\mathbf{c}_1,\mathbf{c}_2)+\tilde{\xi}^2\frac{\partial}{\partial\mathbf{c}_1}\cdot\frac{\partial}{\partial\mathbf{c}_2}\chi_H(c_1)\chi_H(c_2)\right], \end{aligned} \tag{90}$$

where, in the last step, we have taken into account that s_1 is large and we have introduced the term $\frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{\partial}{\partial \mathbf{c}_2} \chi_H(c_1)\chi_H(c_2)$. This term does not contribute to the integral, but is written for convenience, to make a connection with the global correlation function $\phi_H(\mathbf{c}_1, \mathbf{c}_2) \equiv \int d\mathbf{l} h_H(\mathbf{l}, \mathbf{c}_1, \mathbf{c}_2)$ which fulfills Eq. (70). In doing so, we find that the autocorrelation function of $\Pi(\mathbf{k}, s)$ reads

$$\langle \Pi(\mathbf{k}, s_1)\Pi(-\mathbf{k}, s_2)\rangle_H \simeq \frac{\tilde{V}^2}{N} k^2 C_{xy}(s_2 - s_1), \tag{91}$$

where

$$C_{xy}(s_2 - s_1) = \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1x} c_{1y} c_{2x} c_{2y} e^{\Lambda_2(s_2 - s_1)} \phi_H(\mathbf{c}_1, \mathbf{c}_2). \tag{92}$$

As $\phi_H(\mathbf{c}_1, \mathbf{c}_2)$ is the integral of the correlation function $h_H(\mathbf{l}, \mathbf{c}_1, \mathbf{c}_2)$, C_{xy} can be identified as the correlation function of the global quantity $\sum_{i=1}^N V_x(t)V_y(t)$.

C Time integral of the correlation function

In this Appendix we evaluate the time integral of the correlation function $\langle S(\mathbf{k}, 0)\Pi(-\mathbf{k}, s)\rangle_H$ in the hydrodynamic limit. Using the notation of the previous Appendix, we have

$$\begin{aligned}
&\int_0^\infty \mathrm{d}s \langle S(\mathbf{k}, 0)\Pi(-\mathbf{k}, s)\rangle_H \\
&= ik\frac{\tilde{V}^2}{N} \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\perp} c_{2\parallel} c_{2\perp} \left[\frac{e^{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)s}}{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)}\right]_0^\infty Q_2\Gamma(\mathbf{c}_1, \mathbf{c}_2) \\
&= -ik\frac{\tilde{V}^2}{N} \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\perp} c_{2\parallel} c_{2\perp} \frac{1}{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)} Q_2\Gamma(\mathbf{c}_1, \mathbf{c}_2) \\
&= ik\frac{\tilde{V}^2}{N} \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\perp} c_{2\parallel} c_{2\perp} \frac{Q_2[\Lambda(\mathbf{k}, \mathbf{c}_1) + \Lambda(-\mathbf{k}, \mathbf{c}_2)]}{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)} \phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2),
\end{aligned} \tag{93}$$

where we have introduced the function

$$\phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2) = \int \mathrm{d}\mathbf{l} e^{-i\mathbf{k}\cdot\mathbf{l}} h_H(\mathbf{l}, \mathbf{c}_1, \mathbf{c}_2), \tag{94}$$

that fulfills the Fourier transform of Eq. (23)

$$[\Lambda(\mathbf{k}, \mathbf{c}_1) + \Lambda(-\mathbf{k}, \mathbf{c}_2)]\phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2) = -\Gamma(\mathbf{c}_1, \mathbf{c}_2) + \tilde{\xi}^2 \frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{\partial}{\partial \mathbf{c}_2} \chi_H(c_1)\chi_H(c_2)\delta(\mathbf{k}). \tag{95}$$

Note that the last term in the previous equation only appears for $\mathbf{k} = \mathbf{0}$. Taking into account that $c_\perp \chi_H(c)$ is left eigenfunction associated to the null eigenvalue and after some algebra we obtain

$$\begin{aligned}
&\int_0^\infty \mathrm{d}s \langle S(\mathbf{k}, 0)\Pi(-\mathbf{k}, s)\rangle_H \\
&= \frac{\tilde{V}^2}{N} k^2 \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\parallel} c_{1\perp} c_{2\parallel} c_{2\perp} \frac{1}{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)} \phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2) \\
&\quad + \frac{\tilde{V}^2}{N} ik \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\perp} c_{2\parallel} c_{2\perp} \phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2).
\end{aligned} \tag{96}$$

Now, let us consider the hydrodynamic limit of (96). The first term gives

$$\frac{\tilde{V}^2}{N} k^2 \int \mathrm{d}\mathbf{c}_1 \int \mathrm{d}\mathbf{c}_2 c_{1\parallel} c_{1\perp} c_{2\parallel} c_{2\perp} \frac{1}{Q_2\Lambda(-\mathbf{k}, \mathbf{c}_2)} \phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2) \to -\frac{\tilde{V}^2}{N} k^2 \int_0^\infty \mathrm{d}s C_{xy}(s). \tag{97}$$

The second term can be evaluated by using the following expansion in powers of k

$$[\Lambda(\mathbf{c}) - i\mathbf{k}\cdot\mathbf{c}]^{-1} \simeq \Lambda(\mathbf{c})^{-1} + \Lambda(\mathbf{c})^{-1}(i\mathbf{k}\cdot\mathbf{c})\Lambda(\mathbf{c})^{-1}, \tag{98}$$

which yields

$$\begin{aligned}&\frac{\tilde{V}^2}{N} ik \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1\perp} c_{2\|} c_{2\perp} \phi_H(\mathbf{k}, \mathbf{c}_1, \mathbf{c}_2)\\ &\quad \to \frac{\tilde{V}^2}{N} k^2 \int_0^\infty ds C_{xy}(s) - \frac{\tilde{V}^2}{N} k^2 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \int_0^\infty ds e^{\Lambda(\mathbf{c}_2)s} c_{2y} \phi_H(\mathbf{c}_1, \mathbf{c}_2). \end{aligned}\tag{99}$$

Taking into account (97) and (99), we obtain

$$\begin{aligned}&\int_0^\infty ds \langle S(\mathbf{k}, 0)\Pi(-\mathbf{k}, s)\rangle \to -\frac{\tilde{V}^2}{N} k^2 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \int_0^\infty ds e^{\Lambda(\mathbf{c}_2)s} c_{2y} \phi_H(\mathbf{c}_1, \mathbf{c}_2)\\ &\quad = \frac{\tilde{V}^2}{N} k^2 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} c_{2y} \phi_H(\mathbf{c}_1, \mathbf{c}_2). \end{aligned}\tag{100}$$

D k^2 Component of the correlation function of R_w

In this Appendix we evaluate the k^2 component of the correlation function of R_w in terms of the one and two-particle distribution functions, χ_H and χ_2

$$\phi_H(\mathbf{c}_1, \mathbf{c}_2) = \chi_H(c_1)\delta(\mathbf{c}_{12}) + \chi_2(\mathbf{c}_1, \mathbf{c}_2). \tag{101}$$

The first term is

$$\begin{aligned}\int_0^\infty ds C_{xy}(s) &= \int_0^\infty ds \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{1y} c_{2x} c_{2y} e^{s\Lambda(\mathbf{c}_2)} \phi_H(\mathbf{c}_1, \mathbf{c}_2)\\ &= -\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{1y} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} \phi_H(\mathbf{c}_1, \mathbf{c}_2)\\ &= -\int d\mathbf{c} c_x c_y \Lambda(\mathbf{c})^{-1} c_x c_y \chi_H(c)\\ &\quad -\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{1y} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} \chi_2(\mathbf{c}_1, \mathbf{c}_2). \end{aligned}\tag{102}$$

If we do the same in the second term, we have

$$\begin{aligned}&\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} c_{2y} \phi_H(\mathbf{c}_1, \mathbf{c}_2)\\ &\quad = \int d\mathbf{c} c_x c_y \Lambda(\mathbf{c})^{-1} c_x c_y \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1x} c_{2x} c_{2y} \Lambda(\mathbf{c}_2)^{-1} c_{2y} \chi_2(\mathbf{c}_1, \mathbf{c}_2). \end{aligned}\tag{103}$$

It can be seen that the sum of the two terms only depends on the two-particle correlation function and we obtain the k^2 part of Eq. (73).

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