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Energy fluctuations in a randomly driven granular fluid

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We study the behavior of the energy fluctuations in the stationary state of a uniformly heated granular gas. The equation for the one-time two-particle correlation function is derived and the hydrodynamic eigenvalues are identified. Explicit predictions are subsequently determined for energy fluctuations. The results explain Monte Carlo numerical data reported in previous studies [Eur. Phys. J. B **51**, 377 (2006)].

Keywords: energy fluctuations; granular gas

1. Introduction

Recent years have witnessed ongoing interest in the microscopic and macroscopic properties of granular media. In such systems, a simple ingredient—energy dissipation resulting from collisions—has far-reaching consequences [1], with rich phenomenology: non-Gaussian velocity distributions [2,3], non-equipartition of energy [4–7], and spontaneous symmetry breaking [8–10], to name but a few. Theoretically, one of the tools used to understand this body of phenomena is kinetic theory, which is extended naturally to these systems by introducing an 'inelastic collision rule' in which the energy is not conserved. Although most of the work carried out thus far has focused on the one-particle properties, and on the study of the corresponding Boltzmann equation, it has been shown that correlations are also important and, as a matter of fact, necessary to understand the behavior of the system when vortices or cluster are developed [11,12], or even in simpler situations where spatial homogeneity is enforced [13,14].

As a consequence of dissipation in collisions, the total energy of an isolated granular system decays monotonically in time. Under certain conditions, the system reaches a homogeneous cooling state in which the time dependence of the one-particle distribution function is entirely embodied in the kinetic (so-called granular) temperature, which evolves with time as t^{-2} [2,15]. It is experimentally difficult to probe such a regime (see, however, [16,17]). Nevertheless, it is possible to maintain a granular system in the fast-flow regime by injecting energy in such a way that a stationary state is reached. In these states, the energy injected by the thermostat is compensated by the energy dissipated in collisions. Several mechanisms can be introduced in order to obtain a stationary state. If, for example, energy is injected by a moving boundary such as a vibrating piston, the system reaches an inhomogeneous stationary state [18]. In this work, we will focus on a granular gas that is heated uniformly, coupling the velocity of each particle to a white noise, the so-called 'stochastic thermostat' [3,19–27]. For this kind of forcing, which is relevant for some two-dimensional experimental configurations with a rough vibrating piston [28], the system reaches a homogeneous stationary state after a transient regime. The advantage of such a driving mechanism is that it lends itself to theoretical progress. In this context, the single-particle distribution function has been characterized [3] and long-range correlations predicted from a hydrodynamic treatment [21]. More recently, the fluctuations of the total energy have been analysed [29] (see also [30] for a related numerical study in an inhomogeneous system). In Ref. [29], the second moment of the total energy fluctuations was evaluated by neglecting the correlations, which, by and large, could not explain the simulation results. The objective of the present work is to clarify and quantify the influence of the inelasticity induced correlations on the total energy fluctuations. The methods used bear some similarities to those reported in [31], where it was shown that, for

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the unforced system, the contribution coming from the correlations is of the same order as that coming from the one-particle distribution function itself.

The paper is organized as follows. In Section 2, the equation for the two-particle distribution function is derived, taking due account of the thermostat, while in Section 3 the results are particularized to the homogeneous stationary state, which will play the role of our reference state in subsequent analysis. There we also summarize the main results already known and pertaining to the one-particle distribution function. In Section 4 we analyse the hydrodynamic equations for a homogeneous linear perturbation of the reference state, and obtain the corresponding modes and eigenvalues, which are finally used in Section 5 to meet our objective and obtain an explicit expression for the variance of the total energy.

2. Heated granular gas: two-body kinetic description

We consider a gas of N hard disks (dimension $d=2$) or spheres ($d=3$) of mass m and diameter σ that collide inelastically with a coefficient of normal restitution α [1]. The system is heated uniformly by adding a random component to the velocity of each particle at equal times [21,22]. The driving is implemented in such a way that the time between random kicks is small compared with the mean free time. Then, between collisions, the velocities of the particles undergo a large number of kicks due to the thermostat. In addition, we will assume that the 'jump moments' of the velocities of the particles verify

$$\begin{aligned} B_{ij,\beta\gamma} &\equiv \lim_{\Delta t\to 0}\frac{\langle \Delta v_{i,\beta}\Delta v_{j,\gamma}\rangle}{\Delta t} \\ &= \xi_0^2\delta_{ij}\delta_{\beta\gamma}+\frac{\xi_0^2}{N}(\delta_{ij}-1)\delta_{\beta\gamma}, \quad i,j=1,\dots,N, \\ &\quad \beta,\gamma=1,\dots,d, \end{aligned} \tag{1}$$

where we have introduced $\Delta v_{i,\beta}\equiv v_{i,\beta}(t+\Delta t)-v_{i,\beta}(t)$, $v_{i,\beta}(t)$ being the β component of the velocity of particle i at time t. We have also introduced the strength of the noise, ξ_0^2, and $\langle\cdots\rangle$, which denotes the average over different realizations of the noise. The non-diagonal terms (corresponding to $i\neq j$ and $\beta=\gamma$) are necessary in order to conserve the total momentum.

In the dilute limit, assuming molecular chaos, i.e. that no correlations exist *between colliding particles*, and that the sizes of the jumps due to the thermostat are small compared with the *velocity* scale on which the distribution varies, the equation for the single particle distribution function in our system is the Boltzmann–Fokker–Planck equation [3,22]

$$\frac{\partial}{\partial t}f(x_1,t)+L^{(0)}(x_1)f(x_1,t)=J[f|f]+\frac{\xi_0^2}{2}\left(\frac{\partial}{\partial \mathbf{v}_1}\right)^2 f(x_1,t), \tag{2}$$

where x_i is short-hand for position-momenta coordinates $\{\mathbf{r}_i,\mathbf{v}_i\}$ and

$$L^{(0)}(x_1)=\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1}. \tag{3}$$

The inelastic collision operator $J[f|f]$ reads

$$J[f\,|\,f]=\sigma^{d-1}\int d\mathbf{v}_2\,\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2)f_1(\mathbf{r},\mathbf{v}_1,t)f_1(\mathbf{r},\mathbf{v}_2,t), \tag{4}$$

where

$$\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2)=\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})[\alpha^{-2}b_{\hat{\sigma}}^{-1}-1], \tag{5}$$

with $\mathbf{g}=\mathbf{v}_1-\mathbf{v}_2$ the relative velocity, Θ the Heaviside step function, $\hat{\boldsymbol{\sigma}}$ a unit vector joining the centers of the particles at contact and $b_{\hat{\boldsymbol{\sigma}}}^{-1}$ an operator replacing the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ appearing on the right by the precollisional values

$$\mathbf{v}_1^*\equiv b_{\hat{\boldsymbol{\sigma}}}^{-1}\mathbf{v}_1=\mathbf{v}_1-\frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \tag{6}$$

$$\mathbf{v}_2^*\equiv b_{\hat{\boldsymbol{\sigma}}}^{-1}\mathbf{v}_2=\mathbf{v}_2+\frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}. \tag{7}$$

The term

$$\frac{\xi_0^2}{2}\left(\frac{\partial}{\partial \mathbf{v}_1}\right)^2 f(x_1,t)$$

is a diffusive Fokker–Plank term, and is a signature of the external noise.

As we shall be studying fluctuations, it is convenient to introduce the two-particle distribution function, $f_2(x_1,x_2,t)$. The quantity $f_2(x_1,x_2,t)dx_1\,dx_2$ is defined as the number of pairs of particles in which one lies inside the differential volume dx_1 centred in x_1 and, likewise with dx_2, x_2 for the second particle. This definition is easily generalized to higher n-particle distribution functions, $f_n(x_1,\dots,x_n)$. The evolution equation for f_2 is [31,32]

$$\begin{aligned} &\left[\frac{\partial}{\partial t}+L^{(0)}(x_1)+L^{(0)}(x_2)\right]f_2(x_1,x_2,t) \\ &\quad =\delta(\mathbf{r}_{12})\sigma^{d-1}\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2)f_2(x_1,x_2,t) \\ &\qquad +\sigma^{d-1}\int dx_3[\delta(\mathbf{r}_{13})\bar{T}_0(\mathbf{v}_1,\mathbf{v}_3)+\delta(\mathbf{r}_{23})\bar{T}_0(\mathbf{v}_2,\mathbf{v}_3)] \\ &\qquad \times f_3(x_1,x_2,x_3,t)+F_{\mathrm{TH}}, \end{aligned} \tag{8}$$

where we have introduced F_{TH} which accounts for the external driving. The evolution Equation (8) essentially contains three parts: the free streaming on the left-hand side, the two terms on the right-hand side corresponding to collisions, and the last term, F_{TH}, due to the thermostat. The collisional contribution is split into one part corresponding to collisions of particles with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and the other that involves collisions of particles with velocities $\mathbf{v}_1$ or $\mathbf{v}_2$ with a third particle with arbitrary velocity, $\mathbf{v}_3$. The collisional contribution is identical to that appearing in the absence of forcing [31]. We concentrate now on the new term, F_{TH}. Assuming that the sizes of the jumps due to the thermostat are small compared with the scale in which the distribution f_2 varies, we can expand F_{TH} in the spirit of the Fokker–Planck description [33]

$$
\begin{aligned}
F_{\mathrm{TH}} &\simeq \frac{1}{2}\sum_{\beta,\gamma=1}^{d}\sum_{i,j=1}^{2} B_{ij,\beta\gamma}\frac{\partial}{\partial v_{i,\beta}}\frac{\partial}{\partial v_{j,\gamma}} f_2(x_1,x_2,t)\\
&= \frac{1}{2}\xi_0^2\left[\frac{\partial^2}{\partial v_1^2}+\frac{\partial^2}{\partial v_2^2}-\frac{2}{N}\frac{\partial}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{v}_2}\right] f_2(x_1,x_2,t), \quad (9)
\end{aligned}
$$

where we have taken into account Equation (1), and we have explicitly assumed that the jump moments $B_{ij,\beta\gamma}$ do not depend on the magnitude of the velocities of the particles.

Let us introduce the two-particle and three-particle correlation functions through the usual cluster expansion

$$
f_2(x_1,x_2,t) = f_1(x_1,t)f_1(x_2,t) + g_2(x_1,x_2,t), \quad (10)
$$

and

$$
\begin{aligned}
f_3(x_1,x_2,x_3,t) =& f_1(x_1,t)f_1(x_2,t)f_1(x_3,t)\\
&+ g_2(x_1,x_2,t)f_1(x_3,t) + g_2(x_1,x_3,t)f_1(x_2,t)\\
&+ g_2(x_2,x_3,t)f_1(x_1,t) + g_3(x_1,x_2,x_3,t).
\end{aligned} \quad (11)
$$

The equation for the correlation function $g_2(x_1,x_2,t)$ can be obtained following the same lines as in Refs. [31,34]. Neglecting the three-body correlations, g_3, in Equation (8), we obtain

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t} + L^{(0)}(x_1) + L^{(0)}(x_2)\right] g_2(x_1,x_2,t)\\
&\quad = \delta(\mathbf{r}_{12})\sigma^{d-1}\bar{T}_0(\mathbf{v}_1,\mathbf{v}_2)f_1(x_1,t)f_1(x_2,t) + [K(x_1,t)\\
&\qquad + K(x_2,t)]g_2(x_1,x_2,t) - \frac{\xi_0^2}{N}\frac{\partial}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{v}_2} f_1(x_1,t)f_1(x_2,t),
\end{aligned} \quad (12)
$$

where we have introduced the linear operator $K(x_i,t)$ defined as

$$
\begin{aligned}
K(x_i,t) =& \sigma^{d-1}\int \mathrm{d}x_3\, \delta(\mathbf{r}_{i3})\bar{T}_0(\mathbf{v}_i,\mathbf{v}_3)(1+\mathcal{P}_{i3})f_1(x_3,t)\\
&+ \frac{\xi_0^2}{2}\left(\frac{\partial}{\partial \mathbf{v}_i}\right)^2,
\end{aligned} \quad (13)
$$

and where the permutation operator $\mathcal{P}_{ab}$ interchanges the labels of particles a and b in the quantities on which it acts. As will become clear below, the $1/N$ term in Equation (12) is crucial for the calculation of the energy fluctuations.

3. The stationary state

It has been shown numerically that, after a transient time, the system reaches a homogeneous stationary state [21] in which the energy input from the thermostat is compensated by the energy lost in collisions. In this section we will particularize the equations of the previous section to this state, summarizing the results that are already known concerning the one-particle distribution function and that are required for our theoretical analysis.

The Boltzmann–Fokker–Planck Equation (2) for the distribution function, $f_{\mathrm{H}}(\mathbf{v}_1)$, in the stationary homogeneous state is

$$
\frac{\xi_0^2}{2}\left(\frac{\partial}{\partial \mathbf{v}_1}\right)^2 f_{\mathrm{H}}(\mathbf{v}_1) + J[f_{\mathrm{H}}\mid f_{\mathrm{H}}] = 0. \quad (14)
$$

It is convenient to introduce the scaled distribution function χ_{H}

$$
f_{\mathrm{H}}(\mathbf{v}) = \frac{n_{\mathrm{H}}}{v_{\mathrm{H}}^d}\chi_{\mathrm{H}}(c), \quad (15)
$$

where n_{H} is the homogeneous density, $v_{\mathrm{H}} = (2T_{\mathrm{H}}/m)^{1/2}$ is the thermal velocity defined from the granular temperature

$$
T_{\mathrm{H}} = \frac{2}{\mathrm{d}n_{\mathrm{H}}}\int \mathrm{d}\mathbf{v}\frac{1}{2}mv^2 f_{\mathrm{H}}(\mathbf{v}), \quad (16)
$$

and $\mathbf{c} = \mathbf{v}/v_{\mathrm{H}}$ is the rescaled velocity. The distribution function has been studied in Ref. [3], where an approximate expression for $\chi_{\mathrm{H}}(c)$ was derived to second order in Sonine polynomials [35]

$$
\chi_{\mathrm{H}}(c) = \frac{e^{-c^2}}{\pi^{d/2}}\left(1 + a_2(\alpha)S_{d/2-1}^2(c^2)\right), \quad (17)
$$

with

$$
S_{d/2-1}^2(c^2) = \frac{1}{2}c^4 - \frac{1}{2}(d+2)c^2 + \frac{1}{8}d(d+2), \quad (18)
$$

and $a_2(\alpha)$ a coefficient related to the kurtosis of the function $\chi_{\mathrm{H}}(c)$

$$\frac{d}{d+2}\frac{\langle c^4\rangle_{\mathrm{H}}}{\langle c^2\rangle_{\mathrm{H}}^2} = 1 + a_2(\alpha). \tag{19}$$

An approximate expression for a_2 reads (see [23,36] for a discussion of various possible approximations)

$$a_2 = \frac{16(1-\alpha)(1-2\alpha^2)}{73 + 56d - 24\alpha d - 105\alpha + 30(1-\alpha)\alpha^2}. \tag{20}$$

The expression for the temperature in the first Sonine approximation is

$$T_{\mathrm{H}} = m\left[\frac{\mathrm{d}\xi_0^2\sqrt{\pi}}{(1-\alpha^2)\Omega_d n_{\mathrm{H}}\sigma^{d-1}}\right]^{2/3}(1+\mathcal{O}(a_2)), \tag{21}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional solid angle.

We now turn to the equation for the correlation function, $g_{2,\mathrm{H}}(x_1, x_2)$. It is convenient to introduce the rescaled correlation function $\tilde{g}_{\mathrm{H}}$ via

$$g_{2,\mathrm{H}}(x_1, x_2) = \frac{n_{\mathrm{H}}}{\ell^d v_{\mathrm{H}}^{2d}}\tilde{g}_{\mathrm{H}}(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2), \tag{22}$$

where $\ell = (n_{\mathrm{H}}\sigma^{d-1})^{-1}$ is proportional to the mean free path and $\mathbf{l} = \mathbf{r}/\ell$. In these units, the equation for the reduced function $\tilde{g}_{\mathrm{H}}$ reads

$$\begin{aligned}\left[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2) - \mathbf{c}_{12}\cdot\frac{\partial}{\partial\mathbf{l}_{12}}\right]&\tilde{g}_{\mathrm{H}}(\mathbf{l}_{12}, \mathbf{c}_1, \mathbf{c}_2)\\ = -\delta(\mathbf{l}_{12})\bar{T}_0(\mathbf{c}_1, \mathbf{c}_2)&\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)\\ + \tilde{\xi}_0^2\frac{n_{\mathrm{H}}\ell^d}{N}\frac{\partial}{\partial\mathbf{c}_1}\cdot\frac{\partial}{\partial\mathbf{c}_2}&\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2),\end{aligned} \tag{23}$$

where we have introduced the linearized Boltzmann–Fokker–Planck operator $\Lambda(\mathbf{c})$

$$\begin{aligned}\Lambda(\mathbf{c}_i)h(\mathbf{c}_i) = &\int \mathrm{d}\mathbf{c}_3\bar{T}_0(\mathbf{c}_i, \mathbf{c}_3)(1 + P_{i3})\chi_{\mathrm{H}}(c_3)h(\mathbf{c}_i)\\ &+ \frac{\tilde{\xi}_0^2}{2}\left(\frac{\partial}{\partial\mathbf{c}_i}\right)^2 h(\mathbf{c}_i),\end{aligned} \tag{24}$$

with rescaled noise amplitude

$$\tilde{\xi}_0^2 = \frac{\xi_0^2\ell}{v_{\mathrm{H}}^3}. \tag{25}$$

As can be seen from Equation (23), the correlation function, $\tilde{g}_{\mathrm{H}}$, is determined by the properties of the linearized Boltzmann–Fokker–Planck operator, Λ, and by the one-particle distribution function χ_{H}. Consequently, it is important to study the spectral properties of Λ, in particular the upper (hydrodynamic) part of the spectrum, in order to understand the fluctuations of global quantities. In the case of a granular gas in the homogeneous cooling state [37], and for a system under ballistic annihilation dynamics [38], it has been shown that it is possible to find the hydrodynamic eigenvalues and eigenfunctions of the linearized Boltzmann operator. Once these quantities are known, it becomes possible to evaluate the fluctuations of the relevant global quantities in the so-called 'hydrodynamic approximation' [31,34]. Below, we will see that we can evaluate the fluctuations of the total energy in an equivalent approximation, but without knowledge of the eigenfunction associated with the energy. The only information needed is the form of the linearized hydrodynamic equations and, in particular, the eigenvalues.

4. Hydrodynamic equations

4.1. *Evolution of homogeneous perturbations*

The objective in this section is to consider the linearized hydrodynamic equations around a homogeneous perturbation in order to extract information about the behavior of a small perturbation of the total energy.

The complete nonlinear hydrodynamic equations for a granular system heated by a stochastic thermostat are [21,25]

$$\frac{\partial}{\partial t}n = -\nabla\cdot(n\mathbf{u}), \tag{26}$$

$$\frac{\partial}{\partial t}\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{mn}\nabla_j P_{ij}, \tag{27}$$

$$\frac{\partial}{\partial t}T = -\mathbf{u}\cdot\nabla T - \frac{2}{dn}(\nabla\cdot\mathbf{q} + P_{ij}\nabla_j u_i) - \zeta T + m\xi_0^2, \tag{28}$$

where P_{ij} is the pressure tensor, $\mathbf{q}$ is the heat flux and ζ is the cooling rate, which is also a functional of the distribution function

$$\begin{aligned}\zeta = &\frac{(1-\alpha^2)m\pi^{(d-1)/2}\sigma^{d-1}}{4\mathrm{d}\Gamma[(d+3)/2]nk_{\mathrm{B}}T}\\ &\times\int \mathrm{d}\mathbf{v}_1\int \mathrm{d}\mathbf{v}_2\,|\mathbf{v}_1 - \mathbf{v}_2|^3 f(\mathbf{r}, \mathbf{v}_1, t)f(\mathbf{r}, \mathbf{v}_2, t).\end{aligned} \tag{29}$$

Considering a homogeneous state, the previous equations reduce to

$$\frac{\partial}{\partial t}n = 0, \quad \frac{\partial}{\partial t}\mathbf{u} = \mathbf{0}, \quad \frac{\partial}{\partial t}T = -\zeta T + m\xi_0^2. \tag{30}$$

In the long time limit, the system is expected to approach a steady state with a constant temperature given by the equation

$$\zeta_{\mathrm{H}}(f_{\mathrm{H}})T_{\mathrm{H}} = m\xi_0^2. \tag{31}$$

Substituting the explicit form of the one-particle distribution function (15) in the equation above, we obtain the temperature given in Equation (21).

Now let us consider a homogeneous state close to this homogeneous stationary state. We can write the hydrodynamic fields as $n(t) = n_{\mathrm{H}} + \delta n$, $\mathbf{u}(t) = \delta\mathbf{u}$ and $T(t) = T_{\mathrm{H}} + \delta T$. We also define the dimensionless hydrodynamic fields

$$\delta\rho(\tau) = \frac{\delta n}{n_{\mathrm{H}}}, \quad \delta\mathbf{w}(\tau) = \frac{\delta\mathbf{u}}{v_{\mathrm{H}}}, \quad \delta\theta(\tau) = \frac{\delta T}{T_{\mathrm{H}}}, \tag{32}$$

where we have introduced the dimensionless time scale τ, proportional to the number of collisions per particle, defined as

$$\tau = \int_0^t \mathrm{d}t' \, \frac{v_{\mathrm{H}}}{\ell} = \frac{v_{\mathrm{H}}}{\ell} t. \tag{33}$$

Assuming that the deviations are small, and taking into account Equations (30) and (31), we can write the linearized evolution equations for the dimensionless hydrodynamic fields in this new time scale:

$$\frac{\partial}{\partial\tau}\delta\rho = 0, \quad \frac{\partial}{\partial\tau}\delta\mathbf{w} = 0, \quad \frac{\partial}{\partial\tau}\delta\theta = -\zeta_0\delta\rho - \frac{3}{2}\zeta_0\delta\theta, \tag{34}$$

where $\zeta_0 = \ell\zeta_{\mathrm{H}}/v_{\mathrm{H}}$ is a dimensionless coefficient that is a functional of the one-particle distribution function in the stationary state. Its expression in the first Sonine approximation is [3]

$$\zeta_0 = \frac{(16 + 3a_2)\pi^{(d-1)/2}(1-\alpha^2)}{8\sqrt{2}\,d\Gamma(d/2)}. \tag{35}$$

To obtain the equation for $\delta\theta$ we have assumed that the perturbed distribution function scales as

$$f(\mathbf{v}, t) = \frac{n}{\bar{v}(t)^d}\chi_{\mathrm{H}}\left(\frac{\mathbf{v}}{\bar{v}(t)}\right), \tag{36}$$

where $\bar{v}(t) = [2T(t)/m]^{1/2}$, and χ_{H} is the same scaled distribution function as for the reference stationary state. This assumption has already been used and tested numerically in [21]. Then, the cooling rate ζ for the state under scrutiny is proportional to $T^{1/2}(t)$ and we obtain the equation for the linearized energy written above. Equations (34) indicate that a perturbation in the total number of particles or total momentum does not decay, as a consequence of the fact that these variables are conserved, but a perturbation in the total energy will decay (exponentially in τ) to the stationary value, as expected. Moreover, as the equation for the temperature can be rewritten in the following form:

$$\frac{\partial}{\partial\tau}\left(\frac{2}{3}\delta\rho + \delta\theta\right) = -\frac{3}{2}\zeta_0\left(\frac{2}{3}\delta\rho + \delta\theta\right), \tag{37}$$

we can identify the hydrodynamic eigenvalues $\lambda = 0$ and $\gamma = -\frac{3}{2}\zeta_0$, λ being $(d+1)$-fold degenerate. For the sake of clarity, it proves convenient to relabel these eigenvalues as

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\frac{3}{2}\zeta_0, \tag{38}$$

where λ_2 is d-fold degenerate. The associated hydrodynamic modes, $\{y_\beta\}_{\beta=1}^{d+2}$, are

$$y_1 = \delta\rho, \quad \mathbf{y}_2 = \delta\mathbf{w}, \quad y_3 = \frac{2}{3}\delta\rho + \delta\theta. \tag{39}$$

The + sign in the last equation stems from the fact that an increased density leads to enhanced dissipation, and hence a lower temperature.

4.2. *Enforcing consistency with the linearized Boltzmann–Fokker–Planck equation description*

We now turn our attention to the problem of finding the linearized hydrodynamic equations for a homogeneous perturbation, directly from the Boltzmann–Fokker–Planck equation. Enforcing consistency with the macroscopic considerations of Section 4.1, we will infer useful properties on the hydrodynamic part of the spectrum of $\Lambda(\mathbf{c})$. We first introduce the scaled deviation of the distribution function

$$\delta\chi(\mathbf{c}, \tau) = \frac{v_{\mathrm{H}}^d}{n_{\mathrm{H}}}[f(\mathbf{v}, t) - f_{\mathrm{H}}(\mathbf{v})]. \tag{40}$$

The evolution of the scaled distribution is governed by

$$\frac{\partial}{\partial\tau}\delta\chi(\mathbf{c}, \tau) = \Lambda(\mathbf{c})\delta\chi(\mathbf{c}, \tau), \tag{41}$$

where the operator $\Lambda(\mathbf{c})$ is the linearized Boltzmann–Fokker–Planck operator defined in (24). Let us also introduce the scalar product

$$\langle f(\mathbf{c}) \mid g(\mathbf{c})\rangle \equiv \int \mathrm{d}\mathbf{c}\, \chi_{\mathrm{H}}^{-1}(c) f^*(\mathbf{c}) g(\mathbf{c}), \tag{42}$$

where f^* is the complex conjugate of f. Interestingly, the hydrodynamic modes introduced in (39) can then be written as

$$y_\beta = \langle \bar{\xi}_\beta \mid \delta\chi\rangle, \quad \beta = 1, 2, 3, \tag{43}$$

where

$$\bar{\xi}_1(\mathbf{c}) = \chi_H(c), \quad \bar{\boldsymbol{\xi}}_2(\mathbf{c}) = \mathbf{c}\chi_H(c), \quad \bar{\xi}_3(\mathbf{c}) = \left(\frac{c^2}{d} - \frac{1}{6}\right)\chi_H(c). \tag{44}$$

Taking the scalar product of the linearized Boltzmann–Fokker–Planck Equation (41) with the functions $\bar{\xi}_\beta$, we obtain the linear Equations (34) (in the hydrodynamic time scale, that is if we wait long enough so that fast modes have vanished) only if the spectrum of Λ admits the three eigenvalues written in (38), and the associated 'hydrodynamic' eigenfunctions, $\{\xi_\beta\}_{\beta=1}^{d+2}$, obey the orthogonality condition

$$\langle \bar{\xi}_{\beta_1} \mid \xi_{\beta_2} \rangle = \delta_{\beta_1\beta_2}, \quad \beta_1, \beta_2 = 1, 2, 3. \tag{45}$$

In Appendix A, it is shown that the null eigenvalue is $(d+1)$-fold degenerate, and the corresponding eigenfunctions, ξ_1 and $\boldsymbol{\xi}_2$, are determined. Moreover, as a consequence of particle and total momentum conservation in a collision, $\bar{\xi}_1$ and $\bar{\boldsymbol{\xi}}_2$ are the corresponding left eigenfunctions. We were not able to demonstrate that λ_3 is an eigenvalue of Λ, but we have shown explicitly that

$$\langle \bar{\xi}_3 \mid \xi_\beta \rangle = 0, \quad \text{for } \beta = 1, 2. \tag{46}$$

In the following, we will assume that Λ actually admits this third eigenvalue, with an unknown eigenfunction ξ_3. With the help of this assumption, we will see in the next section that it is possible to define a projector in the hydrodynamic subspace, which opens the way for evaluating the variance of the total energy fluctuations.

5. Energy fluctuations

In this section, we study the fluctuations of the global energy for a system in the stationary state. As we are interested in global quantities, it is convenient to define a global correlation function ϕ_H

$$\phi_H(\mathbf{c}_1, \mathbf{c}_2) \equiv \int d\mathbf{r}_{12}\, \tilde{g}_H(\mathbf{r}_{12}, \mathbf{c}_1, \mathbf{c}_2). \tag{47}$$

The energy fluctuations can be written as a functional of this correlation function and the one-particle distribution function, χ_H, as [31,38]

$$\langle (\delta E)^2 \rangle_H = \frac{m^2}{4} N v_H^4 \left[\int d\mathbf{c}\, c^4 \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2\, c_1^2 c_2^2 \phi_H(\mathbf{c}_1, \mathbf{c}_2) \right]. \tag{48}$$

In order to evaluate the integral over the correlation function ϕ_H, we start from (23), integrating over the position variable. Assuming periodic boundary conditions, the spacial gradient terms disappear and we have the following equation for ϕ_H:

$$[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2)]\phi_H(\mathbf{c}_1, \mathbf{c}_2) = \Gamma(\mathbf{c}_1, \mathbf{c}_2), \tag{49}$$

where

$$\Gamma(\mathbf{c}_1, \mathbf{c}_2) = -\bar{T}_0(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2) + \tilde{\xi}_0^2 \frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{\partial}{\partial \mathbf{c}_2} \chi_H(c_1)\chi_H(c_2). \tag{50}$$

The solubility condition for Equation (49) is that Γ does not have components in the subspace associated with the null eigenvalue. In our case, this subspace is generated by $\{\bar{\xi}_1, \bar{\boldsymbol{\xi}}_2\}$. Due to the conservation of the number of particles and total momentum in a collision, and to the symmetry of the second term of Γ, we have

$$\langle \bar{\xi}_1(\mathbf{c}_1)\bar{\boldsymbol{\xi}}_2(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1, \mathbf{c}_2)\rangle = \langle \bar{\boldsymbol{\xi}}_2(\mathbf{c}_1)\bar{\xi}_1(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1, \mathbf{c}_2)\rangle = \mathbf{0}, \tag{51}$$

$$\langle \bar{\xi}_1(\mathbf{c}_1)\bar{\xi}_1(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1, \mathbf{c}_2)\rangle = \langle \bar{\xi}_{2,i}(\mathbf{c}_1)\bar{\xi}_{2,j}(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1, \mathbf{c}_2)\rangle = 0, \tag{52}$$

for $i \neq j$. The case $i = j$ is analysed in Appendix B, where it is shown that $\langle \bar{\xi}_{2,i}(\mathbf{c}_1)\bar{\xi}_{2,i}(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1, \mathbf{c}_2)\rangle = 0$. In order to prove this property, the presence of the second term on the right-hand side of Equation (50) is essential. Hence, the solubility condition holds and the problem of finding ϕ_H with Equation (49) is well defined.

Let us also define a projector P_{12} in the hydrodynamic subspace as

$$P_{12}h(\mathbf{c}_1, \mathbf{c}_2) = \sum_{\beta_1=1}^{3}\sum_{\beta_2=1}^{3} \langle \bar{\xi}_{\beta_1}(\mathbf{c}_1)\bar{\xi}_{\beta_2}(\mathbf{c}_2) \mid h(\mathbf{c}_1, \mathbf{c}_2)\rangle \times \xi_{\beta_1}(\mathbf{c}_1)\xi_{\beta_2}(\mathbf{c}_2), \tag{53}$$

where $\{\xi_\beta\}_{\beta=1}^{3}$ are the right hydrodynamic eigenfunctions of the linearized Boltzmann–Fokker–Planck operator, $\{\bar{\xi}_\beta\}_{\beta=1}^{3}$ the orthogonal set introduced in the previous section, Equation (44), and we have generalized the scalar product by

$$\langle f(\mathbf{c}_1, \mathbf{c}_2) \mid g(\mathbf{c}_1, \mathbf{c}_2)\rangle = \int d\mathbf{c}_1 \int d\mathbf{c}_2\, \chi_H^{-1}(c_1)\chi_H^{-1}(c_2) \times f^*(\mathbf{c}_1, \mathbf{c}_2) g(\mathbf{c}_1, \mathbf{c}_2). \tag{54}$$

Note that P_{12} is a projector even if the function $\bar{\xi}_3$ is not the true left eigenfunction of $\Lambda(\mathbf{c})$ (remember that $\bar{\xi}_1$ and $\bar{\boldsymbol{\xi}}_2$ are the actual left eigenfunctions associated with the null eigenvalue). The fact that the set of functions $\{\bar{\xi}_\beta\}_{\beta=1}^{3}$ and $\{\xi_\beta\}_{\beta=1}^{3}$ fulfil the orthogonality condition (45) is enough to guarantee that $P_{12}^2 = P_{12}$. Using this projector, we define the 'hydrodynamic part' of ϕ_{H} to be the function

$$\phi_{\mathrm{H}}^{(h)}(\mathbf{c}_1,\mathbf{c}_2) \equiv P_{12}\phi_{\mathrm{H}}(\mathbf{c}_1,\mathbf{c}_2) = \sum_{\beta_1=1}^{3}\sum_{\beta_2=1}^{3} a_{\beta_1\beta_2}\xi_{\beta_1}(\mathbf{c}_1)\xi_{\beta_2}(\mathbf{c}_2). \tag{55}$$

The coefficients $a_{\beta_1\beta_2}$ are the quantities we need to evaluate. As they are essentially the first moments of the correlation function ϕ_{H}, they are directly related to the integral we have to calculate in (48). It is tempting to treat ξ_3 as if it was an actual left eigenfunction of the linearized Boltzmann–Fokker–Planck operator, and we will in the following use the approximation

$$P_{12}\Lambda(\mathbf{c}_i) = P_{12}\Lambda(\mathbf{c}_i)P_{12}, \tag{56}$$

which allows us to find a closed equation for $\phi_{\mathrm{H}}^{(h)}$. This approximation has already been invoked in other systems such as the freely evolving granular gas [31], or the probabilistic ballistic annihilation model [34]. With the information available on the linearized Boltzmann–Fokker–Planck operator, it would seem to be the best that can be done technically. Let us also remark that the approximation is exact in the elastic limit. In the inelastic case, some exact results for the fluctuations of the total energy have been obtained for one dimensional Maxwell molecules [40]. Then, applying the projector P_{12} to Equation (49) and taking into account the approximation (56), we obtain the following expressions for the coefficients $a_{\beta_1\beta_2}$:

$$a_{\beta_1\beta_2} = -\frac{\langle\bar{\xi}_{\beta_1}(\mathbf{c}_1)\bar{\xi}_{\beta_2}(\mathbf{c}_2) \mid \bar{T}_0(\mathbf{c}_1,\mathbf{c}_2)\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)\rangle}{\lambda_{\beta_1}+\lambda_{\beta_2}}, \tag{57}$$

where it has been assumed that $\lambda_{\beta_1}+\lambda_{\beta_2}\neq 0$. The coefficients associated with the vanishing eigenvalue cannot be calculated using Equation (49), but are fixed by the boundary conditions. The coefficients $a_{\beta_1\beta_2}$ are evaluated in Appendix C. The expression for $\phi_{\mathrm{H}}^{(h)}$ is finally given by

$$\begin{aligned}\phi_{\mathrm{H}}^{(h)} &= a_{11}\xi_1(\mathbf{c}_1)\xi_1(\mathbf{c}_2) + a_{13}[\xi_1(\mathbf{c}_1)\xi_3(\mathbf{c}_2)+\xi_3(\mathbf{c}_1)\xi_1(\mathbf{c}_2)]\\ &\quad + a_{33}\xi_3(\mathbf{c}_1)\xi_3(\mathbf{c}_2),\end{aligned} \tag{58}$$

where $a_{11}=-1$, $a_{13}=-\frac{1}{3}$ and a_{33} can be obtained as a functional of the one-particle distribution function

$$a_{33} = \frac{\langle\bar{\xi}_3(\mathbf{c}_1)\bar{\xi}_3(\mathbf{c}_2) \mid \bar{T}_0(\mathbf{c}_1,\mathbf{c}_2)\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)\rangle}{3\zeta_0}. \tag{59}$$

An approximate expression is derived in Appendix C, and reads

$$a_{33} = \frac{\left\{\begin{array}{c}-15+7d+14d^2-3(-9+d(9+2d))\alpha\\ +30(1+d)\alpha^2-6(9+d)\alpha^3)\end{array}\right\}}{9d(-19+2d(-7+3\alpha)+3\alpha(9+2(-1+\alpha)\alpha))}. \tag{60}$$

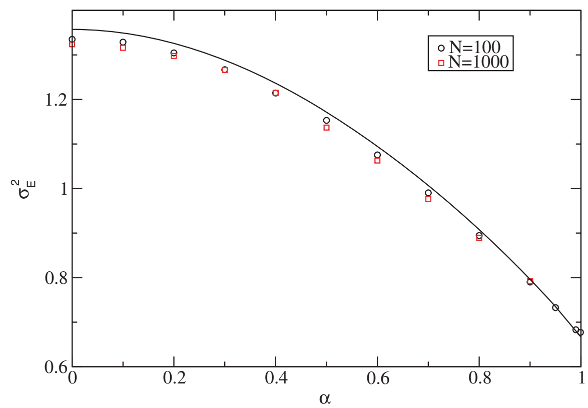

Figure 1. Scaled second moment of the energy fluctuations σ_E^2 as a function of the restitution coefficient α. The solid line is the theoretical prediction and symbols are the two-dimensional Monte Carlo simulation results of Ref. [29].

Taking into account (48) and (58), the variance of the energy fluctuations can finally be written as

$$\sigma_E^2 = N\frac{\langle(\delta E)^2\rangle_{\mathrm{H}}}{\langle E\rangle_{\mathrm{H}}^2} = (a_2+1)\frac{d(d+2)}{4} + d^2 a_{33} - d^2\frac{5}{36}. \tag{61}$$

In Ref. [29], the value of σ_E^2 has been measured by means of the Direct Monte Carlo simulation method (DSMC). In Figure 1 we compare our theoretical prediction (solid line given by (61)) with the DMSC simulations results (symbols). The agreement is satisfactory for the whole range of inelasticities, at variance with the theoretical attempt put forward in [29], which neglected velocity correlations. In particular, we note the non-trivial result in the elastic limit $\lim_{\alpha\to 1^-}\sigma_E^2(\alpha) = d/3$ (i.e. 2/3 in Figure 1, which is well obeyed), while in the free cooling regime, this quantity vanishes [31]. We emphasize that the elastic limit is singular: the behaviour for elastic systems with $\alpha=1$ is not approached by taking the quasi-elastic limit $\alpha\to 1^-$ (we note that the divergence of the different moments of the velocity distribution as $\alpha\to 1$ is nevertheless indicative of the absence of a stationary state when $\alpha=1$). Such a singularity has already been reported in one dimension [39], but, to our knowledge, not for two-dimensional granular systems. It is also interesting to note that the singular nature of the quasi-elastic limit appears at the two-body level through the energy fluctuations, while, as far as rescaled distribution functions are concerned, the one-body level of description is regular, with a well-behaved velocity distribution approaching a Gaussian form [22].

6. Conclusions

The problem of the fluctuations of the total energy of a granular (inelastic) gas maintained in a non-equilibrium stationary state by a random acceleration has been addressed. A numerical study of this quantity has been performed by means of Monte Carlo simulations and an argument assuming an uncorrelated non-Gaussian individual distribution function had been proposed in [29], without success. The main goal of this work was therefore to take due account of velocity correlations in order to study these fluctuations.

To this end, the standard description at the single-particle level is not sufficient, and the two-particle correlation function is needed. We have derived the evolution equation for such an object, and particularized the analysis to the homogeneous stationary state that is reached by the system in the long time limit. Our work shows that this equation is not a straightforward generalization of its counterpart arising in the context of the undriven granular gas (i.e. by only changing the linearized Boltzmann–Fokker–Planck operator into its driven form). A non-trivial non-diagonal term appears in the Fokker–Planck equation for the two-particle distribution function (contribution $\partial_{\mathbf{v}_1} \cdot \partial_{\mathbf{v}_2}$ in Equation (9)), as a consequence of the coupling between velocities due to momentum conservation.

We have seen that, for our purposes, exact knowledge of the hydrodynamic eigenfunctions is not needed. The important point is that we can construct a set of functions $\{\bar{\xi}_\beta\}_{\beta=1}^3$, which are linear combinations of 1, $\mathbf{c}$ and c^2, that are orthogonal to the right eigenfunctions $\{\xi_\beta\}_{\beta=1}^3$ of the linearized Boltzmann–Fokker–Planck operator Λ. This orthogonality property holds for the 'real' left eigenfunctions, that in our case correspond to the null eigenvalue (i.e. density and velocity fields associated with conserved quantities). The function $\bar{\xi}_3$ is not a left eigenfunction of Λ but it can be proved to be orthogonal to ξ_1 and $\boldsymbol{\xi}_2$. In a subsequent step, the linear hydrodynamic equations around the reference state are derived and from that knowledge, the hydrodynamic eigenvalues are identified and the variance of energy fluctuations subsequently derived.

Finally, our prediction has been successfully tested against the numerical results obtained by the Direct Monte Carlo simulation method for the complete range of values of the coefficient of normal restitution α. This provides strong support for the theory developed here and assesses in retrospect the validity of our assumptions.

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References

[1] A. Barrat, E. Trizac, and M.H. Ernst, J. Phys.: Condens. Matter **17**, S2429 (2005).
[2] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).
[3] T.P.C. van Noije and M.H. Ernst, Granular Matter **1**, 57 (1998).
[4] R.D. Wildman and D.J. Parker, Phys. Rev. Lett. **88**, 064301 (2002).
[5] K. Feitosa and N. Menon, Phys. Rev. Lett. **88**, 198301 (2002).
[6] A. Barrat and E. Trizac, Granular Matter **4**, 57 (2002).
[7] A. Santos and J.W. Dufty, Phys. Rev. Lett. **97**, 058001 (2006).
[8] H.J. Schlichtting and V. Nordmeier, MNU **49**, 323 (1996).
[9] J.J. Brey, F. Moreno, R. García-Rojo, and M.J. Ruiz-Montero, Phys. Rev. E **65**, 011305 (2002).
[10] A. Barrat and E. Trizac, Mol. Phys. **101**, 1713 (2003).
[11] T.P.C. van Noije, M.H. Ernst, R. Brito, and J.A.G. Orza, Phys. Rev. Lett. **79**, 411 (1997).
[12] J.J. Brey, F. Moreno, and M.J. Ruiz-Montero, Phys. Fluids **10**, 2965 (1998).
[13] J.J. Brey, M.I. García de Soria, P. Maynar, and M.J. Ruiz-Montero, Phys. Rev. Lett. **94**, 098001 (2005).
[14] J.J. Brey, A. Domínguez, M.I. García de Soria, and P. Maynar, Phys. Rev. Lett. **96**, 158002 (2006).
[15] P.K. Haff, J. Fluid Mech. **134**, 401 (1983).
[16] B. Painter, M. Dutt, and R. Behringer, Physica D **175**, 43 (2003).
[17] C.C. Maaß, N. Isert, G. Maret, and C.M. Aegerter, Phys. Rev. Lett. **100**, 248001 (2008).
[18] J.J. Brey, M.J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **62**, 5339 (2000).
[19] D.R.M. Williams and F.C. MacKintosh, Phys. Rev. E **54**, R9 (1996).
[20] A. Puglisi, V. Loreto, U.M.B. Marconi, and A. Vulpiani, Phys. Rev. E **59**, 5582 (1999).
[21] T.P.C. van Noije, M.H. Ernst, E. Trizac, and I. Pagonabarraga, Phys. Rev. E **59**, 4326 (1999).
[22] I. Pagonabarraga, E. Trizac, T.P.C. van Noije, and M.H. Ernst, Phys. Rev. E **65**, 011303 (2002).
[23] J.M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).
[24] S.J. Moon, M.D. Shattuck, and J.B. Swift, Phys. Rev. E **64**, 031303 (2001).

[25] V. Garzó and J.M. Montanero, Physica A **313**, 336 (2002).
[26] A. Puglisi, P. Visco, A. Barrat, E. Trizac, and F. van Wijland, Phys. Rev. Lett. **95**, 110202 (2005).
[27] M.H. Ernst, E. Trizac, and A. Barrat, J. Statist. Phys. **124**, 549 (2006).
[28] A. Prevost, D.A. Egolf, and J.S. Urbach, Phys. Rev. Lett. **89**, 084301 (2002).
[29] P. Visco, A. Puglisi, A. Barrat, F. van Wijland, and E. Trizac, Eur. Phys. J. B **51**, 377 (2006).
[30] S. Aumaître, J. Farago, S. Fauve, and S. McNamara, Eur. Phys. J. B **42**, 255 (2004).
[31] J.J. Brey, M.I. García de Soria, P. Maynar, and M.J. Ruiz-Montero, Phys. Rev. E **70**, 011302 (2004).
[32] M.H. Ernst and E.G.D. Cohen, J. Statist. Phys. **25**, 153 (1981).
[33] N.G. van Kampen, *Stochastic Proccesses in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[34] P. Maynar, M.I. García de Soria, G. Schehr, A. Barrat, and E. Trizac, Phys. Rev. E **77**, 051127 (2008).
[35] L. Landau and E. Lifshitz, *Physical Kinetics* (Pergamon Press, Oxford, 1981).
[36] F. Coppex, M. Droz, J. Piasecki, and E. Trizac, Physica A **329**, 114 (2003).
[37] J.J. Brey, J.W. Dufty, and M.J. Ruiz-Montero, in *Granular Gas Dynamics*, edited by T. Poeschel and N. Brilliantov (Springer, Berlin, 2003).
[38] M.I. García de Soria, P. Maynar, G. Schehr, A. Barrat, and E. Trizac, Phys. Rev. E **77**, 051128 (2008).
[39] A. Barrat, T. Biben, Z. Racz, E. Trizac, and F. van Wijland, J. Phys. A **35**, 463 (2002); A. Barrat, E. Trizac, and M.H. Ernst, J. Phys. A **40**, 4057 (2007).
[40] G. Costantini, U.M.B. Marconi, and A. Puglisi, J. Stat. Mech.: Theory and Experiments P08031 (2007).

Appendix A. Eigenvalue problem for Λ

We consider here the eigenvalue problem for the homogeneous linear Boltzmann–Fokker–Planck operator Λ, defined in (24)

$$\Lambda(\mathbf{c})\xi_\beta(\mathbf{c}) = \lambda_\beta \xi_\beta(\mathbf{c}). \tag{A1}$$

We are interested in the eigenfunctions and eigenvalues associated with linear hydrodynamics and, to perform the analysis, techniques similar to those in [31,37,38] will be required.

Consider first the function

$$\psi_1(\mathbf{c}) = \chi_{\mathrm{H}}(c). \tag{A2}$$

When the linearized operator Λ acts on χ_{H}, we have

$$\Lambda(\mathbf{c}_1)\chi_{\mathrm{H}}(\mathbf{c}_1) = \int d\mathbf{c}_2 \bar{T}_0(\mathbf{c}_2, \mathbf{c}_3)(1+\mathcal{P}_{12})\chi_{\mathrm{H}}(c_2)\chi_{\mathrm{H}}(c_1) + \frac{\widetilde{\xi}_0^2}{2}\left(\frac{\partial}{\partial \mathbf{c}_1}\right)^2 \chi_{\mathrm{H}}(c_1). \tag{A3}$$

Taking into account the equation for χ_{H}, Equation (14), we obtain the following relation:

$$\Lambda(\mathbf{c}_1)\psi_1(c_1) = -\frac{\widetilde{\xi}_0^2}{2}\left(\frac{\partial}{\partial \mathbf{c}_1}\right)^2 \chi_{\mathrm{H}}(c_1). \tag{A4}$$

Now let us considerer the function

$$\psi_2(\mathbf{c}) = -\frac{\partial}{\partial \mathbf{c}}\chi_{\mathrm{H}}(c). \tag{A5}$$

Taking the derivate in the equation obeyed by $\chi_{\mathrm{H}}(\mathbf{c}-\mathbf{w})$ with respect to $\mathbf{w}$, and subsequently evaluating the result for $\mathbf{w}=0$, we obtain

$$\Lambda(\mathbf{c}_1)\psi_2(\mathbf{c}_1) = \mathbf{0}. \tag{A6}$$

Finally, we will consider the function

$$\psi_3(\mathbf{c}) = \mathbf{c}\cdot\frac{\partial}{\partial \mathbf{c}}\chi_{\mathrm{H}}(c). \tag{A7}$$

From the equation obeyed by $\psi_3(\lambda\mathbf{c}_1)$, we can take the derivate with respect to λ, and evaluate the result for $\lambda=1$. We arrive at an equation for $\psi_3(\mathbf{c}_1)$,

$$\Lambda(\mathbf{c}_1)\psi_3(\mathbf{c}_1) = (d+3)\frac{\widetilde{\xi}_0^2}{2}\left(\frac{\partial}{\partial \mathbf{c}_1}\right)^2 \chi_{\mathrm{H}}(c_1). \tag{A8}$$

From Equations (A4), (A6) and (A8), we can identify two eigenfunctions of Λ. Making use of (A4) and (A8), it appears that

$$\Lambda(\mathbf{c})\left(\frac{1}{3}\frac{\partial}{\partial \mathbf{c}}\cdot[\mathbf{c}\chi_{\mathrm{H}}(c)] + \chi_{\mathrm{H}}(c)\right) = 0. \tag{A9}$$

Hence, from Equations (A6) and (A9) we can conclude that the null eigenvalue is $(d+1)$-fold degenerate with the eigenfunctions

$$\xi_1(\mathbf{c}) = \frac{1}{3}\frac{\partial}{\partial \mathbf{c}}\cdot[\mathbf{c}\chi_{\mathrm{H}}(c)] + \chi_{\mathrm{H}}(c), \quad \xi_2 = -\frac{\partial}{\partial \mathbf{c}}\chi_{\mathrm{H}}(c). \tag{A10}$$

Appendix B. Evaluation of the coefficient $a_{2,i2,i}$

In this appendix, we show that $\langle \bar{\xi}_{2,i}(\mathbf{c}_1)\bar{\xi}_{2,i}(\mathbf{c}_2) \mid \Gamma(\mathbf{c}_1,\mathbf{c}_2)\rangle = 0$. The integral corresponding to the second term of Γ is simply

$$\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1,i} c_{2,i} \widetilde{\xi}_0^2 \frac{\partial}{\partial \mathbf{c}_1}\cdot\frac{\partial}{\partial \mathbf{c}_2}\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2) = \widetilde{\xi}_0^2. \tag{B1}$$

The other term can be written as

$$\begin{aligned}
&\int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1,i}c_{2,i}\bar{T}_0(\mathbf{c}_1,\mathbf{c}_2)\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2) \\
&= \int d\mathbf{c}_1 \int d\mathbf{c}_2 \chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})[b_{\hat{\boldsymbol{\sigma}}}-1]c_{1,i}c_{2,i} \\
&= \int d\mathbf{c}_1 \int d\mathbf{c}_2 \chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12}) \\
&\quad\times\left[(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})\frac{1+\alpha}{2}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})c_{12,i}\hat{\sigma}_i - \frac{(1+\alpha)^2}{4}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{c}_{12})^2\hat{\sigma}_i^2\right] \\
&= \frac{\pi^{(d-1)/2}}{\Gamma[(d+3)/2]}\frac{1-\alpha^2}{4d}\int d\mathbf{c}_1\int d\mathbf{c}_2\,\chi_{\mathrm{H}}(c_1)\chi_{\mathrm{H}}(c_2)c_{12}^3 = \widetilde{\xi}_0^2,
\end{aligned} \tag{B2}$$

which is the desired result.

Appendix C. Evaluation of the coefficients $a_{\beta\beta'}$

In this appendix we evaluate the coefficients $a_{\beta\beta'}$. As the number of particles and the total momentum are conserved quantities in our system, we have

$$\langle(\delta N)^2\rangle = 0, \quad \langle\delta P_i \delta P_j\rangle = 0, \tag{C1}$$

$$\langle\delta N \delta P_i\rangle = 0, \quad \langle\delta N \delta E\rangle = 0, \tag{C2}$$

$$\langle\delta E \delta P_i\rangle = 0. \tag{C3}$$

Enforcing the above constraints, we obtain

$$\int d\mathbf{c}\chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 \phi_H(\mathbf{c}_1, \mathbf{c}_2) = 1 + a_{11} = 0, \tag{C4}$$

$$\int d\mathbf{c} c_i \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} \phi_H(\mathbf{c}_1, \mathbf{c}_2) = a_{12} = 0, \tag{C5}$$

$$\int d\mathbf{c} c^2 \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 \phi_H(\mathbf{c}_1, \mathbf{c}_2) = \frac{d}{2} + d a_{13} + \frac{d}{6} a_{11} = 0, \tag{C6}$$

$$\int d\mathbf{c} c_i c_j \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{2j} \phi_H(\mathbf{c}_1, \mathbf{c}_2) = \tfrac{1}{2}\delta_{ij} + a_{2i2j} = 0, \tag{C7}$$

$$\int d\mathbf{c}\, c_i c^2 \chi_H(c) + \int d\mathbf{c}_1 \int d\mathbf{c}_2\, c_{1i} c_2^2 \phi_H(\mathbf{c}_1, \mathbf{c}_2) = a_{23} = 0. \tag{C8}$$

As a consequence, the values of some coefficients follow

$$a_{11} = -1, \quad a_{12} = 0,$$

$$a_{13} = -\frac{1}{3}, \quad a_{2i2j} = -\tfrac{1}{2}\delta_{ij}, \quad a_{23} = 0. \tag{C9}$$

Of course, the coefficients associated with $\lambda_{\beta_1} + \lambda_{\beta_1} \neq 0$ could also have been calculated directly by Equation (57), obtaining the same results. The coefficient a_{33} is evaluated using (57) and it can be written in terms of the one-particle distribution function as

$$\begin{aligned} a_{33} &= \frac{\langle \bar{\xi}_3(\mathbf{c}_1)\bar{\xi}_3(\mathbf{c}_2) \mid \bar{T}_0(\mathbf{c}_1, \mathbf{c}_2)\chi_H(c_1)\chi_H(c_2)\rangle}{3\zeta_0} \\ &= \frac{1}{18} + \frac{b(\alpha)}{3\zeta_0}, \end{aligned} \tag{C10}$$

where

$$\begin{aligned} b(\alpha) = &-\frac{\pi^{(d-1)/2}}{\Gamma[(d+5)/2]d^2} \\ &\times \int d\mathbf{c}_1 \int d\mathbf{c}_2 \chi_H(c_1)\chi_H(c_2)\vartheta(\mathbf{c}_1, \mathbf{c}_2), \end{aligned} \tag{C11}$$

with

$$\begin{aligned} \vartheta(\mathbf{c}_1, \mathbf{c}_2) = &\frac{(1-\alpha^2)(d+1+2\alpha^2)}{16} c_{12}^5 \\ &+ \frac{(d+5) - \alpha^2(d+1) + 4\alpha}{4} c_{12}^3 C^2 \\ &- \frac{1+\alpha}{2}(2d+3-3\alpha)c_{12}(\mathbf{C}\cdot\mathbf{c}_{12})^2, \end{aligned} \tag{C12}$$

and $\mathbf{C} = (\mathbf{c}_1 + \mathbf{c}_2)/2$. The coefficient $b(\alpha)$ can be evaluated using the expression of $\chi_H(c)$ in the first Sonine approximation, Equation (17), which yields

$$\begin{aligned} b(\alpha) = &\frac{\left\{ \begin{array}{c} (1+d)(3+d)(2d(a_2 + 16(-1+\alpha) + 15a_2\alpha) \\ +16(-1+\alpha)(-1+2\alpha^2) + a_2(7+3\alpha(-13 \\ +10(-1+\alpha)\alpha)))\Gamma[(1+d)/2] \end{array} \right\}}{\left\{ \begin{array}{c} 2^{1/2}\pi^{d+1/2} 128 d^2(-2+(5+d)/2) \\ (-1+(5+d)/2)\Gamma[d/2]\Gamma[-2+(5+d)/2] \end{array} \right\}} \\ &\times (1+\alpha). \end{aligned} \tag{C13}$$

If we take into account the explicit form of ζ_0 and a_2, given in Equations (20) and (35), respectively, we obtain after some algebra

$$a_{33} = \frac{\left\{ \begin{array}{c} -15 + 7d + 14d^2 - 3(-9 + d(9+2d))\alpha \\ +30(1+d)\alpha^2 - 6(9+d)\alpha^3) \end{array} \right\}}{\left\{ 9d(-19 + 2d(-7+3\alpha) + 3\alpha(9 + 2(-1+\alpha)\alpha)) \right\}}. \tag{C14}$$