Paris-Sud / Paris-Saclay University
M1 General Physics

Year 2016-2017
STATISTICAL PHYSICS
Partial exam 2017

2 HOUR 15 MINUTES
DOCUMENTS, POCKET CALCULATORS AND ANY ELECTRONIC DEVICE NOT ALLOWED
Concise but explicative answers expected throughout. No bonus for verboseness.

A Basic questions

1) What is an order parameter?

2) How are phase transition classified ?

3) What are the main problems faced by a statistical mechanics treatment of phase transitions?

4) What is the meaning of the term *mean-field* and what is the interest of such a view?

5) In which sense can magnets and liquids be viewed as alike ?

6) In a magnetic system, how is c_B, the specific heat at fixed magnetic field, related to the free energy F and the temperature T ?

7) What are liquid crystals ? For studying the isotropic to nematic phase transition of liquid crystals *confined in a planar membrane*, what is the relevant order parameter?

8) Do phase transitions always result from a tradeoff between energy and entropy ?

9) What is called the fluctuation-response connection ?

10) What did Rudolf Peierls achieve in the context of phase transitions?

B Ising magnets / approximate and exact treatments

1) In a mean-field treatment of Ising model, with aligning coupling $J > 0$, on a lattice such that each spin is coupled to z nearest neighbors, what is the phase transition taking place without a magnetic field, and what is the corresponding critical temperature ? How do these results depend on space dimension ?

2) Within mean-field again, how is the magnetization m related to the external magnetic field B and temperature T ? Compute the critical exponent γ governing the behaviour of the susceptibility, for T below the critical temperature.

3) We now restrict to the one-dimensional Ising model with periodic boundary conditions. The N-spin Hamiltonian reads

$$H = -J \sum_{i=1}^{N-1} S_i S_{i+1} - B \sum_{i=1}^{N} S_i \tag{1}$$

We seek for a calculation of the free energy by a transfer matrix technique. Write two distinct and eligible transfer matrices, and check explicitly that they lead to the same partition function. Hint: there are infinitely many ways of writing the transfer matrix. Only two are asked for.

C Landau theory and tricriticality

Within a Landau approach, the free energy is expanded in powers of the order parameter ϕ, in the form

$$\mathcal{R} = \frac{1}{2}\, a_2\, \phi^2 + \frac{1}{4}\, a_4\, \phi^4 + \frac{1}{6}\, a_6\, \phi^6 \,, \tag{2}$$

where in the vicinity of T_c, the coefficient a_2 linearly depends on temperature T: $a_2 = (T - T_c)\, \widetilde{a}_2$ (with $\widetilde{a}_2 > 0$). For simplicity, we suppose that a_4 and a_6 are independent on temperature.

C.1 Order of transitions and sign of coefficients

1. Provide an example of a physical system where such an expansion may be relevant.
2. What should the sign of a_6 be?
3. Sketch the free energy profiles $\mathcal{R}(\phi)$ for different temperatures, treating separately the cases $a_4 > 0$ and $a_4 < 0$. For each case, what is the order of the corresponding phase transition embodied in Eq. (2)?
4. In the first order transition case, we denote T^* the critical temperature. What are the conditions on $\mathcal{R}$ which determine this temperature? At $T = T^*$, give the values of ϕ that minimize the free energy and show that

   $$T^* = T_c + \alpha\, \frac{a_4^2}{\widetilde{a}_2\, a_6} \,.$$

   What is the value of α?
5. Draw a schematic phase diagram in the plane (T, a_4), for fixed values of $\widetilde{a}_2$ and a_6. Indicate the phase transition lines of first and second orders. In which point of the diagram do they meet?

C.2 Study of the tricritical point

We will now assume that $a_4 = 0$, which defines for our model a so-called tricritical point.

1. What is the order of the phase transition ?
2. How does the order parameter depend on temperature, in the vicinity of T_c? Infer from this behaviour the value of the β exponent, defined by $\phi \propto (T_c - T)^\beta$ below the critical point. What is the value of β when $a_4 > 0$?
3. Under the action of an applied external field B, which additional term should appear in the free energy ? What is the exponent δ which measures, at $T = T_c$, the response to B through $\phi \propto B^{1/\delta}$? Same question when $a_4 > 0$.

C.3 Beyond Landau...

We again address the case $a_4 = 0$.

1. Write a free energy functional of the Ginzburg-Landau type, that generalizes expression (2) to situations where ϕ changes with position $\vec{r}$, and which includes a term stemming from an external field $B(\vec{r})$.
2. What is the differential equation fulfilled by $\Gamma(\vec{r}, \vec{r}\,')$ the correlation function of the order parameter? What form does it take when the system is spatially homogeneous?
3. Compute the value of critical exponent ν, dealing with the correlation length. How does it compare to its counterpart when a_4 is non-vanishing?