M1 GP / Partial exam Feb 2019

1- Basic questions

1°) $S(U,V,N)$ entropy

$$dU = TdS - PdV + \mu dN \quad ; \quad U = \text{int energy}$$

$F = U - TS$, Helmholtz free energy

$$dF = -SdT - PdV + \mu dN$$

$H = U + PV$, enthalpy

$$dH = TdS + VdP + \mu dN$$

$G = U + PV - TS$, Gibbs free energy

$$dG = -SdT + VdP + \mu dN$$

2°) Mean-field arguably the simplest

3°) For two hamiltonians depending on same microscopic variables: H and H_0

$$F \leqslant F_0 + \langle H - H_0 \rangle_0$$

↳ variational treatment.

4°) $\# \text{ intensive param} = 2 + \underbrace{c}_{\rightarrow \# \text{ constituents}} - \underbrace{\varphi}_{\hookrightarrow \# \text{ phases}}$

For $c = 2$ (binary) $\rightarrow 4 - \varphi$

$c = 3 \rightarrow 5 - \varphi$

5°)

6°) they express the same physical principles either in terms of pressure [P(V), Maxwell] or free energy [F(V), double tangent]

7°) $c_v = \left.\dfrac{\partial U}{\partial T}\right|_{V(N)}$; $\delta Q = c_v dT + \ell dV = TdS$

$$\Rightarrow dU = c_v dT + (\ell - P) dV$$

$$\Rightarrow c_v = T\left.\frac{\partial S}{\partial T}\right|_V = -T\left.\frac{\partial^2 F}{\partial T^2}\right|_V$$

For c_p, ⚠, $c_p \neq \left.\dfrac{\partial U}{\partial T}\right|_P$

$$\delta Q = c_p dT + h\,dP = TdS$$

$$\Rightarrow c_p = T\left.\frac{\partial S}{\partial T}\right|_P = -T\left.\frac{\partial^2 G}{\partial T^2}\right|_P$$

8°) If we accept the fact that, for those exponents that are defined both below and above T_c, the same quantity applies:

macro: α, β, γ, δ

micro: ν, η

Otherwise, we have $\alpha_+, \alpha_-, \beta, \gamma_+, \gamma_-, \delta$ and ν_+, ν_-, η

9°) $X \rightsquigarrow g(0,\sigma)$; $\langle e^{kX} \rangle = e^{\cancel{k\langle x \rangle}} e^{\frac{1}{2}k^2\sigma^2}$

With $\sigma = 3$: $\langle e^X \rangle = \langle e^{-X} \rangle = e^{9/2}$

$\langle X \rangle = 0$; $\langle X^2 \rangle = 3^2$; $\langle X^3 \rangle = 0$

$\langle X^4 \rangle = 3 \langle X^2 \rangle^2 = 243$

If $X \rightsquigarrow g(1,3)$; $\langle e^X \rangle = e^1 e^{9/2} = e^{11/2}$

$\langle e^{-X} \rangle = e^{-1} e^{9/2} = e^{7/2}$

2- Fluctuation-response for ideal magnets

1°) $\mathcal{H} = -B\sum_{i=1}^{N} S_i$

2°) $Z = \sum_{\mathcal{C}} e^{\beta B \sum S_i} = \sum_{\mathcal{C}} \prod_{i=1}^{N} e^{\beta B S_i}$

$$\boxed{Z = \left[2(\cosh \beta B)\right]^N}$$

3°) $\langle M\rangle = \frac{1}{Z}\sum_{\mathcal{C}} \left(\sum S_i\right) e^{-\beta \mathcal{H}} = \frac{\partial \ln Z}{\partial \beta B}$

$= N\tanh(\beta B)$

4°) $\chi = \frac{\partial \langle M\rangle}{\partial B} = \beta N\left(1-\tanh^2 \beta B\right)$

5°) $\langle M^2\rangle = \frac{1}{Z}\sum_{\mathcal{C}} \left(\sum S_i\right)^2 e^{-\beta \mathcal{H}} = \frac{1}{Z}\frac{\partial^2 Z}{\partial(\beta B)^2}$

$= \frac{1}{(\cosh\beta B)^N}\frac{\partial}{\partial \beta B}\left[N(\cosh\beta B)^{N-1}\sinh(\beta B)\right]$

$= N(N-1)\tanh^2(\beta B) + N$

$\frac{\langle M^2\rangle}{N^2} \xrightarrow[N\to\infty]{} \tanh^2(\beta B)$

6°) $\langle M^2\rangle - \langle M\rangle^2 = N - N\tanh^2(\beta B) = \chi kT$

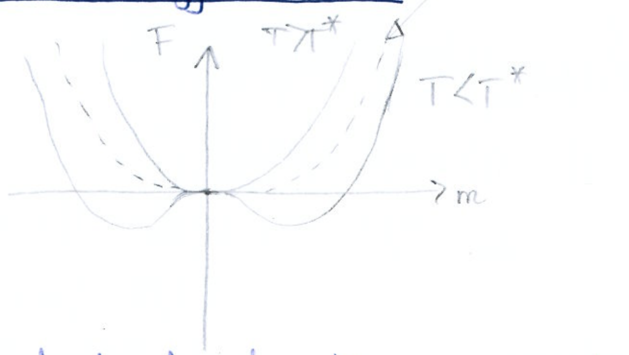
7°) No, interactions are needed

8°) the relation in question 3

9°) See the lectures for: $\boxed{\chi kT = \langle M^2\rangle - \langle M\rangle^2}$

3- Landau vs Bragg-Williams

1°)



2°) Second order phase transition since m continuous

$T_c = T^*$

3°) $\frac{\partial F}{\partial m} = 0 \Rightarrow a_2 m + 4a_4 m^3 = 0$

$m^2 = -\frac{a_2}{4a_4} = \frac{\tilde{a}_2}{4a_4}(T^* - T)$

$\Rightarrow \boxed{\beta = \frac{1}{2}}$

4°) $a_2 m + 4a_4 m^3 = B$ and $\frac{\partial}{\partial B}$

$\hookrightarrow a_2\chi + 4a_4 3m^2\chi =$

$T > T^* \Rightarrow m = 0$ and $\chi = \frac{1}{a_2} \propto \frac{1}{T-T^*}$

$T < T^* \Rightarrow 4m^2 a_4 = -a_2$ and $\chi = -\frac{1}{2a_2} \propto \frac{1}{T^*-T}$

thus $\boxed{\gamma = 1}$ and $\gamma_+ = \gamma_- = \gamma$

5°) $a_2 m + 4a_4 m^3 = B$; $B \to 0$, $m \to 0$

and $a_2 = 0$ for $T = T^*$ $\Rightarrow$ $m^3 \propto B$; $\boxed{\delta = 3}$

6°) From $H = -J\sum_{\langle i,j\rangle} S_i S_j - B\sum S_i$, z is # nearest neighbour

$F = -\frac{JNz}{2}m^2 + NkT\left(\frac{1+m}{2}\right)\log\left(\frac{1+m}{2}\right) + NkT\left(\frac{1-m}{2}\right)\log\left(\frac{1-m}{2}\right) - BNm$

7°) Taylor expansion for $m \to 0$

8°) they are the same.