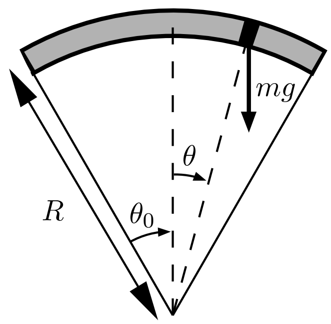
Paris-Sud / Paris-Saclay University
M1 General Physics

Year 2018-2019
STATISTICAL PHYSICS
Final exam 2019

Should be written on a separate paper.
Concise but explicative answers expected throughout. No bonus for verboseness

Symmetry breaking in a mechanical model for phase transitions

An airtight hollow tube, shown in grey on the figure, is bent in a circular arc of radius R with section S. In the tube, there is a movable but gastight piston on which gravitational force mg acts. The piston separates two



chambers (left/right) each containing N ideal gas molecules. Its motion is frictionless. The angular position of the piston is θ. The tube is closed at both ends so that $-\theta_0 \leqslant \theta \leqslant \theta_0$. The volume on both sides of the piston read $SR\,(\theta_0 - \theta)$ and $SR\,(\theta_0 + \theta)$. The gravitational energy of the gas will be neglected throughout; temperature T is uniform and fixed.

1) What is the pressure of the gas in each chamber ? Write the mechanical equilibrium position condition for the piston position, in the form

$$\sin\theta = \frac{T}{T_c}\, f(\theta), \tag{1}$$

where T_c is some constant and f some function to be specified, chosen such that $f(\theta)/\theta \to 1$ for $\theta \to 0$. Check that your results do not depend on S.

2) Sketch a graph of $f(\theta)$ for $-\theta_0 \leqslant \theta \leqslant \theta_0$.

3) For both $T \leqslant T_c$ and $T \geqslant T_c$, plot schematically the solutions to Eq. (1), that we shall denote θ_{eq}.

4) We assume here that T is close to T_c so that $|\theta| \ll \theta_0$ and $f(\theta)$ can be linearized. Show then that $|\theta_{\text{eq}}| \propto |T - T_c|^{\beta}$. For which case $T \leqslant T_c$ or $T \geqslant T_c$ is this behaviour observed ? What is β?

5) The mechanical setup studied here can be viewed as a metaphor for a phase transition; in which system and of which type? What would the order parameter be in the present case? Discuss the analogy.

6) We wish to discuss the stability of the $\theta = 0$ solution. Write the equation of motion of the piston as a differential equation for θ. Conclude on the stability.

7) Our goal next is to recover previous results from thermodynamics considerations. What is the internal energy and the entropy of the gas (assumed monoatomic, although that is not essential)?

8) Assuming again that $|\theta| \ll \theta_0$, write the free energy of the total system (gas and piston), as

$$F(\theta, T) = F_0(T) + \frac{a_2}{2}\,\theta^2 + \frac{a_4}{4}\,\theta^4 + \mathcal{O}(\theta^6). \tag{2}$$

What are a_2 and a_4? How are they related to the results of question 6? Recover T_c, β, and the stability condition of the $\theta = 0$ solution from this analysis.

9) For arbitrary (non small θ), compute $\dfrac{\partial^2 F}{\partial \theta^2}$ for $\theta = 0$. Recover the above stability condition.

10) By a means which is not specified here, a constant force is applied to the piston, tangentially to its direction of motion, so that Eq. (1) becomes

$$\sin\theta = \frac{T}{T_c} f(\theta) - h, \quad (3)$$

where h is constant and does not depend on angular position θ. Discuss the situation $T < T_c$, when h is changed, sweeping negative to positive values. It is convenient here to consider the free energy $F(\theta)$, for several values of h.

11) How would it be possible to define a susceptibility ?

12) We assume that the gas is no longer ideal. The left chamber contains $N(1-\epsilon)$ molecules while the right has $N(1+\epsilon)$, with $\epsilon > 0$. At a certain temperature, a droplet of liquid is found in the right chamber, coexisting with its vapor. In which physical state is the left chamber (vapor, liquid...)?

Reference: *An Exactly Solvable Model Exhibiting a Landau Phase Transition*,
R. Alben, American Journal of Physics **40**, 3 (1972).