Symmetry breaking in a mechanical model of ΦT

1°) Force balance on the piston, involving

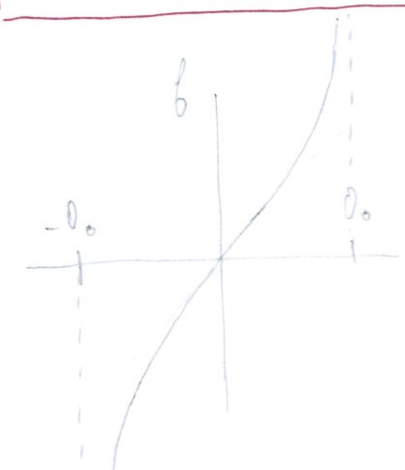
$$P_{Left} = \frac{NkT}{SR(\theta_0+\theta)} \; ; \; P_{Right} = \frac{NkT}{SR(\theta_0-\theta)}$$

$$\Rightarrow mg\sin\theta = \frac{NkT}{\cancel{S}R}\left(\frac{1}{\theta_0-\theta} - \frac{1}{\theta_0+\theta}\right)\cancel{S}$$

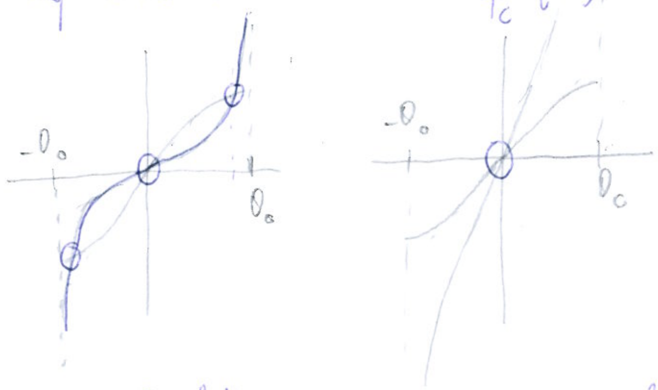
$$\sin\theta = \frac{2NkT}{Rmg}\,\frac{\theta}{\theta_0^2-\theta^2} = \underbrace{\frac{2NkT}{Rmg\theta_0^2}}_{T/T_c}\,\frac{\theta}{1-(\theta/\theta_0)^2}$$

$$\boxed{T_c = \frac{Rmg\theta_0^2}{2Nk} \; , \; f(\theta) = \frac{\theta}{1-\theta^2/\theta_0^2}}$$

2°)



3°) θ_{eq} is solution to $\sin\theta = \frac{T}{T_c} f(\theta)$.



$T<T_c$, 3 solutions
$0, \pm\theta_{eq}$

$T>T_c$, 1 solution
$\theta_{eq}=0$

4°) $|\theta| \ll \theta_0 \Rightarrow f(\theta) \sim \theta$, and we have to solve

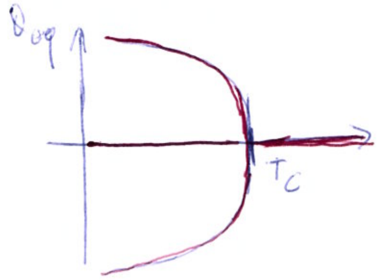
$$\sin\theta \sim \theta - \frac{\theta^3}{3!} \sim \frac{T}{T_c}\theta$$

$$\Rightarrow \theta\left(-\frac{T}{T_c}+1\right) \sim \frac{\theta^3}{6}$$

The non-vanishing solution obeys:

$$\left(1-\frac{T}{T_c}\right) \propto \theta_{eq}^2$$

$$|\theta_{eq}| \propto \sqrt{1-\frac{T}{T_c}} \; , \; T<T_c$$



$$\boxed{\beta = \frac{1}{2}}$$

5°) Similar to 2^{nd} order transition in ferromagnets. The order param is θ_{eq}

6°) $$mR\ddot{\theta} = mg\sin\theta - \frac{NkT}{R}\left(\frac{1}{\theta_0-\theta} - \frac{1}{\theta_0+\theta}\right)$$

$$\boxed{\frac{R}{g}\ddot{\theta} = \sin\theta - \frac{T}{T_c}f(\theta)}$$

For small θ $\frac{R}{g}\ddot{\theta} \sim \theta - \theta\frac{T}{T_c}$

$$\hookrightarrow \ddot{\theta} + \frac{g}{R}\left(-1+\frac{T}{T_c}\right)\theta = 0$$

Stable for $T>T_c$, unstable for $T<T_c$

7°) $U = \frac{3}{2}NkT \times 2$ if monoatomic

Entropy for gas (N,V,T)?

$$Z = \frac{1}{N!}\left(\frac{V}{\Lambda^3}\right)^N \; , \quad \Lambda = \text{de Broglie wave length}$$

$$F = -kT\ln Z \quad ; \ln N! \sim N\log N - N$$

$$= -kTN\log\left(\frac{V}{\Lambda^3}\right) + kTN\log N - NkT$$

$$= kTN\log\left(\frac{N\Lambda^3}{V}\right) - NkT = U-TS$$

$$\Rightarrow TS = -kTN\log\left(\frac{N\Lambda^3}{V}\right) - \frac{5}{2}NkT$$

For an system:

$$F = mgR\cos\theta + NkT\log\left[\frac{N\Lambda^3}{S^2R(\theta_0-\theta)(\theta_0+\theta)}\right] - 2NkT$$

$$\boxed{F = F_0^*(T) + mgR\cos\theta - NkT\log\left(1-\frac{\theta^2}{\theta_0^2}\right)}$$

8) For $|\theta| \ll \theta_0$, $\log(1-x) \simeq -x - \frac{x^2}{2}$

$$F = F_0^*(T) + mgR - mgR\frac{\theta^2}{2} + NkT\left(\frac{\theta^2}{\theta_0^2} + \cancel{\frac{\theta^4}{2\theta_0^4}}\right) + mgR\frac{\theta^4}{4!} + O(\theta^6)$$

negligible

$$F = F_0^*(T) + mgR + \underbrace{\left(2\frac{NkT}{\theta_0^2} - mgR\right)}_{a_2 = mgR\left(\frac{T}{T_c}-1\right)}\frac{\theta^2}{2} + \underbrace{\frac{1}{6}mgR}\frac{\theta^4}{4}$$

$$a_4 = \frac{1}{6}mgR + \frac{2NkT}{\theta_0^4}$$

We recover equilibria at $\theta = 0$ and at $0 = a_2\theta + a_4\theta^3 \Rightarrow \theta_{eq} = \pm\sqrt{-\frac{a_2}{a_4}}$

$$\theta_{eq} \propto \pm\sqrt{\left(1-\frac{T}{T_c}\right)}$$

$\theta = 0$ is unstable for $T < T_c$; $\beta = \frac{1}{2}$

9) $$\frac{\partial^2 F}{\partial\theta^2} = -mgR\cos\theta + NkT\frac{\partial}{\partial\theta}\left(\frac{2\theta/\theta_0^2}{1-\theta^2/\theta_0^2}\right)$$

$$= -mgR\cos\theta + \frac{2NkT}{\theta_0^2}\left[\frac{1}{1-\theta^2/\theta_0^2} - \theta\frac{2\theta/\theta_0^2}{(1-\theta^2/\theta_0^2)^2}\right]$$

$$\boxed{\left.\frac{\partial^2F}{\partial\theta^2}\right|_{\theta=0} = -mgR\cos\theta + \frac{2NkT}{\theta_0^2}}$$

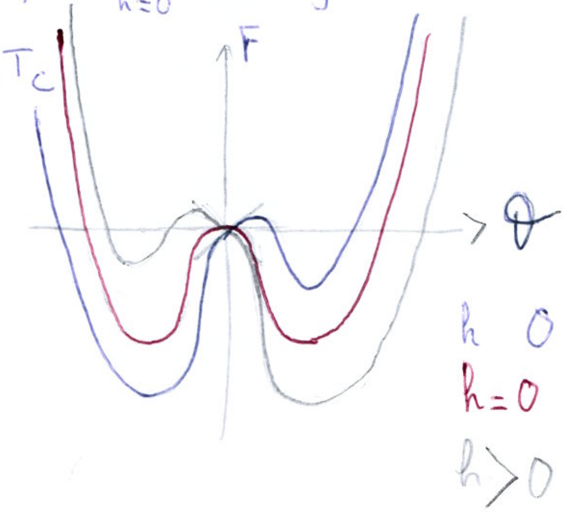
$= a_2$

NB Eq of motion: $MR\ddot{\theta} = -\frac{1}{R}\frac{\partial F}{\partial\theta}$; $\boxed{M\ddot{\vec{r}} = -\vec{\nabla}_r F}$

10) this corresponds to a force $F = h\,mgR$, independent from θ

$\Rightarrow F(\theta) = F_{h=0} - mgRh\theta$

For $T < T_c$



Changing h from $h<0$ where $\theta_{eq}<0$ to $h>0$ where $\theta_{eq}>0$, there is a discontinuous jump of θ

↳ scenario of a first order phase transi

11) We can monitor how θ changes with h → define then $\Delta\theta_{eq} = \theta_{new} - \theta_{h=0}$

Then $\chi = \frac{\Delta\theta}{h}$ for h small

or $\chi = \frac{\partial\Delta\theta_{eq}}{\partial h} = \frac{\partial\theta_{eq}}{\partial h}$

12) 

$N_{right} > N_{left} \Rightarrow \theta_{eq} < 0$

$\Rightarrow P_{left} > P_{right}$

If gas/liquid coexist at (T, P_{right}), then only liquid can be found at $P_{left} > P_{right}$

→ LIQUID STATE