M1 GP . Partial exam ; 02/2020

1 - Basic questions : see lectures

3°) When rod-like molecules lie in a plane, we denote θ the angle wrt the director $\hat{n}$ and define order param

$$S = \frac{2\langle \cos^2\theta \rangle - 1}{2-1} = 2\langle \cos^2\theta\rangle - 1$$

4°) Non interacting spins:

$$H = -B\sum_{i=1}^{N} S_i$$

$$Z = \sum_{S_1 \cdots S_N} e^{-\beta H} = \left[2\cosh(\beta B)\right]^N$$

$$M = \sum_{i=1}^{N} S_i \; ; \; \langle M\rangle = \frac{1}{Z}\sum_{S_1 \cdots S_N} M\, e^{\beta B M}$$

$$\langle M\rangle = \frac{\partial}{\partial(\beta B)}\log Z = -\frac{\partial F}{\partial B}\Big|_T$$

$$\boxed{\langle M\rangle = N \tanh(\beta B)}$$

5°) $\langle M\rangle = -\frac{\partial F}{\partial B}\Big|_T$, see above ; $F(T,B)$

6°) $C_B = T\frac{\partial S}{\partial T}\Big|_B$; $dF = -SdT - MdB$

$$C_B = -T\frac{\partial^2 F}{\partial T^2}\Big|_B$$

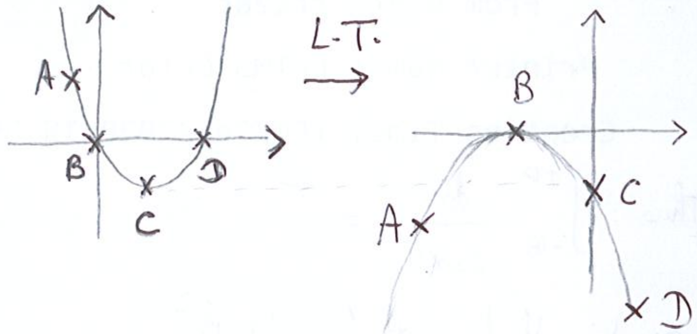
7°) $C_M = T\frac{\partial S}{\partial T}\Big|_M$; $d\tilde{F} = d(F + BM)$

$$= -SdT + BdM$$

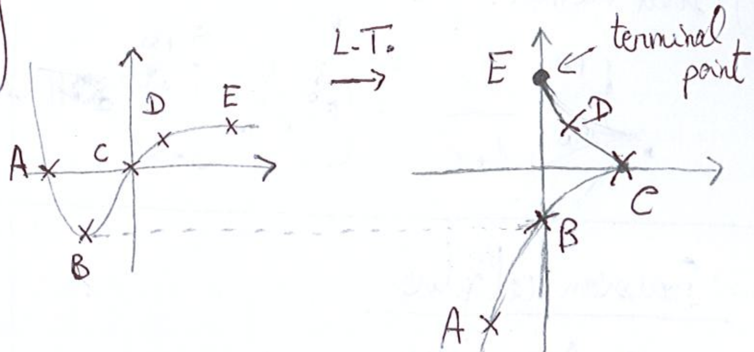
$$C_M = -T\frac{\partial^2 \tilde{F}}{\partial T^2}\Big|_M \; ; \; \tilde{F}(T,M)$$

$$\tilde{F} = F + BM$$

8°)

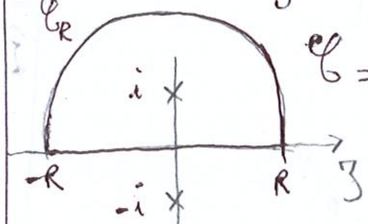


9°)



this latter graph is not that of a function ; due to lack of convexity of the input

10°) a) $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ has poles $\pm i$



$\mathcal{C} = [-R,R] \cup \mathcal{C}_R$ upper semi-circle radius R

Residue theorem: $\oint_{\mathcal{C}} f(z)dz = 2i\pi \times \text{residue}$

$$= 2i\pi\,\frac{1}{2i} = \pi$$

$$\int_{-R}^{R} f + \underbrace{\int_{\mathcal{C}_R} f}_{\text{can be bound}} = \pi$$

can be bound ; $|z^2+1| \geqslant R^2 - 1 \geqslant \frac{R^2}{2}$

if R large enough

$$\Rightarrow \left|\int_{\mathcal{C}_R} f(z)dz\right| \leqslant \left|\int_{\mathcal{C}_R}\frac{dz}{|1+z^2|}\right|$$

$$\leqslant \frac{2}{R^2}\pi R \xrightarrow[R\to\infty]{} 0$$

Thus : $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi$

f) Direct method: $th' = 1 + th^2$

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \left[\operatorname{argth} x \right]_{-\infty}^{+\infty} = \pi$$

2- Gaussian calculus

$x \rightsquigarrow g(m, \sigma)$

1°) $\langle e^x \rangle = e^m \, e^{\sigma^2/2}$

$$e^{\langle x \rangle} = e^m < \langle e^x \rangle$$

Inequality always true for a convex up function

2°) $\langle e^{-x} \rangle = e^{-m} \, e^{\sigma^2/2}$

3°) $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$; $\frac{\partial p}{\partial x} = -\frac{x}{\sigma^2} p(x)$

$$\left\langle \frac{\partial f}{\partial x} \right\rangle = \int \frac{\partial f}{\partial x} p(x) dx$$

$$= \left[f(x) p(x) \right]_{-\infty}^{+\infty} + \frac{1}{\sigma^2} \int x f(x) p(x) dx$$

$$\Rightarrow \boxed{\langle x f(x) \rangle = \sigma^2 \left\langle \frac{\partial f}{\partial x} \right\rangle}$$

4°) Take $f(x) = x^3$: $\langle x^4 \rangle = 3\sigma^4 = 3\langle x^2 \rangle^2$

$f(x) = x^5$: $\langle x^6 \rangle = 5\sigma^2 \langle x^4 \rangle$
$= 5 \times 3 \times \sigma^6$

5°) Hence $\langle x^n \rangle = 0$ for n odd

$\langle x^n \rangle = (n-1)(n-3) \cdots 1$ for n even

6°) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\langle e^x \rangle = \sum_{p=0}^{\infty} \frac{\langle x^{2p} \rangle}{(2p)!} =$$

$$= \sum_p \frac{(2p-1)(2p-3) \cdots 1}{(2p)!} \sigma^{2p}$$

$$= \sum_p \frac{\sigma^{2p}}{(2p)(2p-2) \cdots 2}$$

$$= \sum_p \frac{(\sigma^2/2)^p}{p!} = e^{\sigma^2/2} \text{ ; OK}$$

3- Magnetic domain walls

BC can be enforced by a strong negative B field for $x \to -\infty$ / positive for $x \to +\infty$.

Extremality : $F = \int \mathcal{L}\left(m, \frac{dm}{dx}\right) dx$

$$\frac{\partial \mathcal{L}}{\partial m} = \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{dm}{dx}\right)} \right)$$

$$\Rightarrow \boxed{-m + m^3 = \frac{d^2 m}{dx^2}}$$

Solve... yields m^*

$$\Rightarrow \left(\frac{dm}{dx}\right)^2 = \frac{m^4}{2} - m^2 + \underbrace{C}_{\frac{1}{2}} = \frac{1}{2}(m^2 - 1)^2$$

$$\frac{dm}{dx} = \frac{1}{\sqrt{2}} (1 - m^2)$$

$$\Rightarrow \boxed{m^*(x) = th\left(\frac{x}{\sqrt{2}}\right)}$$