M1 General Physics
Year 2019-2020
STATISTICAL PHYSICS
Final exam 2020

Should be written on a separate paper.
Concise but explicative answers expected throughout. No bonus for verboseness

The Potts model

We consider a variant of Ising model, the so-called Potts model, where the N degrees of freedom (spins) $\sigma_1, \ldots, \sigma_N$ may each take q different values, $\sigma_i \in \{1, \ldots, q\}$, where q is an arbitrary integer ≥ 2. The spins interact with the following Hamiltonian

$$H(\sigma_1, \ldots, \sigma_N) = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j} - \sum_{i=1}^{N} h_{\sigma_i} \, , \tag{1}$$

where $J \geqslant 0$ is the coupling constant and the summation above runs through pairs of nearest neighbors on some lattice, δ denotes the Kronecker symbol (giving one if the two arguments are equal, 0 otherwise) and $h_1, \ldots, h_q$ can be seen as magnetic fields acting on the q possible values of the spins. Such an approach is useful to study theoretically a variety of "hard" or "soft" condensed matter problems in a unified framework, and is also relevant to model some experimental systems (such as atomic adsorption, magnetic coarsening, foams, bubbles etc.).

In the remainder, we shall work in the canonical ensemble, the system being in equilibrium at temperature T. We denote the Boltzmann constant by k_{B} and averages over the Boltzmann-Gibbs distribution by $\langle \cdot \rangle$; $\beta = 1/(k_B T)$.

A. Qualitative questions

1) If Potts model has to exhibit any connection to the standard spin 1/2 Ising model, for which value of q should it be, and why (qualitative answer only, no calculation) ?

2) If there is a critical temperature in Potts model, what do you expect it to be (no calculation)?

3) What do you expect for the large T behaviour?

B. Limiting cases, order parameter and connection to Ising model

4) When all fields $h_\mu = 0$ ($\mu = 1, \ldots q$), describe a possible ground state (minimizing the Hamiltonian). How many ground states are there?

5) We assume that $h_1 > 0$ while all other fields are vanishing. How many ground state are there? Describe them.

6) Same question for $h_1 > h_2 > 0$ while all other fields vanish.

7) What if all fields $h_\mu \neq 0$ (all being different)?

8) We assume $h_1 > 0$ while all other fields vanish. We denote by $\langle x \rangle$ the mean fraction of spins in state 1 (*i.e.* such that $\sigma_i = 1$).

a) What do you expect for $\langle x \rangle$ when $h_1 \to 0$? When $h_1 \to \infty$?

b) Which of the above situations can we qualify as ordered, and which as disordered?

c) From the above limiting behaviors, what order parameter m can you construct with $\langle x \rangle$, that would go from 0 in a disordered situation to 1 under perfect order?

9) We consider here the $q = 2$ case.

a) Making use of the identities

$$\delta_{\sigma_i^{(\mathrm{I})},\sigma_j^{(\mathrm{I})}} = \frac{1 + \sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})}}{2}\,, \qquad \delta_{\sigma_i^{(\mathrm{I})},+1} = \frac{1 + \sigma_i^{(\mathrm{I})}}{2}\,, \qquad \delta_{\sigma_i^{(\mathrm{I})},-1} = \frac{1 - \sigma_i^{(\mathrm{I})}}{2}, \tag{2}$$

establish that Potts model is then equivalent, up to a constant energy, to Ising model with variables $\sigma_i^{(\mathrm{I})} = \pm 1$ and a Hamiltonian

$$H(\sigma_1^{(\mathrm{I})}, \ldots, \sigma_N^{(\mathrm{I})}) = -\sum_{i,j=1}^{N} J_{i,j}^{(\mathrm{I})} \sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})} - h^{(\mathrm{I})} \sum_{i=1}^{N} \sigma_i^{(\mathrm{I})}\,. \tag{3}$$

Express $J_{i,j}^{(\mathrm{I})}$ and $h^{(\mathrm{I})}$ as functions of parameters pertaining to Potts model.

b) Do you expect a phase transition for $q = 2$ and if so, of what type?

C. The one-dimensional setting: transfer matrix

The N spins live here on a chain, with periodic boundary conditions ($\sigma_1 = \sigma_{N+1}$). There is no external field and the Hamiltonian reads

$$H(\sigma_1, \ldots, \sigma_N) \;=\; -J \sum_{i=1}^{N} \delta_{\sigma_i,\sigma_{i+1}} \tag{4}$$

10) Write the definition of the partition function Z, without attempting to compute it.

11) Rewrite Z introducing a $q \times q$ transfer matrix $\mathbb{T}$. Define $\mathbb{T}$ explicitly for the case $q = 3$. Hint: this matrix is symmetric; its entries can take only two possible values.

12) Here $q = 3$ unless otherwise stated. Show that $\mathbb{T}$ admits ${}^t(1,1,1)$ as an eigenvector, where ^{t} denotes the transpose. What is the corresponding eigenvalue t_+? The other eigenvalue, t_-, is degenerate. What is its expression? Write the corresponding eigenvectors. You can use the fact that they have to be perpendicular to the previous one, and search for them in the form ${}^t(0,?,?)$.

13) Compute the partition function.

14) Write the free energy per spin in the thermodynamic limit. What do you conclude in terms of phase transition?

15) How do results generalize to arbitrary q?

16) Explain the idea for obtaining the correlation length ξ in the model (no calculation asked).

17) The explicit calculation of the correlation length yields

$$\xi \;\propto\; \frac{1}{\log(t_+/t_-)}. \tag{5}$$

What conclusion can be drawn from this relation? What is the expected behaviour of ξ with temperature? Is it compatible with the result in Eq. (5) ?