The Potts model - correction

A. Qualitative questions

1) In this case, any Ising spin can be in one of two states, so that a connection presumably requires $q = 2$.

2) We should have kT_c on the order of the relevant energy scale in the model, that is J. This leaves unspecified the q dependence. We expect T_c to increase with q, to balance enhanced order.

3) When T is large, all q states are equally populated.

B. Limiting cases, order parameter and connection to Ising model

4) When all fields $h_\mu = 0$ ($\mu = 1, \dots q$), all spins align to the same value; the ground state is q-fold degenerate.

5) All spins should be in state 1. There is a unique ground state.

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8) If all fields $h_\mu \neq 0$, we have to find the largest, that will "pin" the system, and lead to a unique ground state.

a) Here $h_1 > 0$ while all other fields vanish. When $h_1 \to 0$, we have $\langle x \rangle \to 1/q$, while when h_1 becomes large, we will get $\langle x \rangle \to 1$.

b) The disordered situation is for $h_1 \to 0$ while large h_1 leads to order (all spins alike).

c) We therefore propose the order parameter

$$\boxed{m = \frac{q\langle x \rangle - 1}{q - 1}}. \tag{1}$$

9) We assume $q = 2$.

a) One can associate states $\sigma^{(\mathrm{I})} = +1$ to $\sigma = 1$ and $\sigma^{(\mathrm{I})} = -1$ to $\sigma = 2$. Making use of the identities

$$\delta_{\sigma_i^{(\mathrm{I})},\sigma_j^{(\mathrm{I})}} = \frac{1 + \sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})}}{2}, \qquad \delta_{\sigma_i^{(\mathrm{I})},+1} = \frac{1 + \sigma_i^{(\mathrm{I})}}{2}, \qquad \delta_{\sigma_i^{(\mathrm{I})},-1} = \frac{1 - \sigma_i^{(\mathrm{I})}}{2} \tag{2}$$

we get

$$H = -\sum_{i,j=1}^{N} J_{i,j}\frac{1 + \sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})}}{2} - h_1\sum_{i=1}^{N}\frac{1 + \sigma_i^{(\mathrm{I})}}{2} - h_2\sum_{i=1}^{N}\frac{1 - \sigma_i^{(\mathrm{I})}}{2} \tag{3}$$

$$= -\sum_{i,j=1}^{N}\frac{J_{i,j}}{2}\sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})} - \frac{h_1 - h_2}{2}\sum_{i=1}^{N}\sigma_i^{(\mathrm{I})} - \left[\frac{1}{2}\sum_{i,j=1}^{N}J_{i,j} + N\frac{h_1 + h_2}{2}\right]. \tag{4}$$

By identification :

$$H(\sigma_1^{(\mathrm{I})}, \dots, \sigma_N^{(\mathrm{I})}) = -\sum_{i,j=1}^{N} J_{i,j}^{(\mathrm{I})}\sigma_i^{(\mathrm{I})}\sigma_j^{(\mathrm{I})} - h^{(\mathrm{I})}\sum_{i=1}^{N}\sigma_i^{(\mathrm{I})} \quad \text{with} \quad \boxed{J_{i,j}^{(\mathrm{I})} = \frac{J_{i,j}}{2} \quad \text{and} \quad h^{(\mathrm{I})} = \frac{h_1 - h_2}{2}} \tag{5}$$

The square bracket in (4) is an immaterial constant.

b) For $q = 2$ we thus expect a second order phase transition.

C. The one-dimensional setting : transfer matrix and renormalization

10) With

$$H(\sigma_1, \ldots, \sigma_N) \;=\; -J \sum_{i=1}^{N} \delta_{\sigma_i,\sigma_{i+1}} \tag{6}$$

the partition function is

$$Z \;=\; \sum_{\sigma_1,\sigma_2,\ldots\sigma_N} \prod_{i=1}^{N} \exp\left(\beta J \delta_{\sigma_i,\sigma_{i+1}}\right) \tag{7}$$

11) Introducing the $q \times q$ transfer matrix $\mathbb{T}$ such that

$$\mathbb{T}(\sigma_i, \sigma_j) \;=\; \exp\left(\beta J \delta_{\sigma_i,\sigma_j}\right) \tag{8}$$

we can write

$$Z \;=\; \mathrm{Tr}(\mathbb{T}^N) \tag{9}$$

For the case $q = 3$, this gives :

$$\mathbb{T} \;=\; \begin{pmatrix} e^{\beta J} & 1 & 1 \\ 1 & e^{\beta J} & 1 \\ 1 & 1 & e^{\beta J} \end{pmatrix} \tag{10}$$

For $q > 3$, the structure is the same, with exponential terms on the diagonal, and 1 on every non-diagonal entry.

12) $\mathbb{T}$ is a so-called circulant matrix, and therefore simple to diagonalize. It is seen that $\mathbb{T}$ admits the eigenvector $|+\rangle = {}^t(1,1,1)$, with eigenvalue $t_+ = e^{\beta J} + 2$. The other eigenvalue is two-fold degenerate. Since we know the trace, we readily find that its value is $t_- = e^{\beta J} - 1$. The two associated eigenvectors, which have to be perpendicular to $|+\rangle$ are ${}^t(1, -1/2, -1/2)$ and ${}^t(-1/2, 1, -1/2)$. Note that $t_- < t_+$. Another possibly more convenient choice is to take these eigenvectors as ${}^t(0, 1, -1)/\sqrt{2}$ and ${}^t(0, -1, 1)/\sqrt{2}$.

In the general case,

$$\boxed{t_+ = e^{\beta J} + q - 1 \quad , \qquad t_- = e^{\beta J} - 1}. \tag{11}$$

13) The eigenvalues being know, the trace of $\mathbb{T}^N$ follows :

$$Z \;=\; t_+^N \;+\; 2\, t_-^N \;=\; \left(e^{\beta J} + 2\right)^N \;+\; 2\, \left(e^{\beta J} - 1\right)^N \tag{12}$$

14) In the thermodynamic limit, the free energy per spin is

$$\beta f \;=\; -\log\left(e^{\beta J} + 2.\right) \tag{13}$$

This expression is analytic in T ; there is no phase transition, which is expected (one dimensional model with short range interactions).

15) The results generalize to arbitrary q :

$$t_+ \;=\; e^{\beta J} + q - 1 \;, \quad t_- = e^{\beta J} - 1 \;, \quad \boxed{Z \;=\; \left(e^{\beta J} + q - 1\right)^N \;+\; (q-1)\, \left(e^{\beta J} - 1\right)^N}. \tag{14}$$

16) See the transfer matrix procedure for the Ising model, seen during the tutorials.

17) The correlation length is finite at all temperatures ; there is no phase transition. Yet, it appears that $\xi \to \infty$ when $T \to 0$, so that we may consider that the system exhibits a transition strictly at $T = 0$.