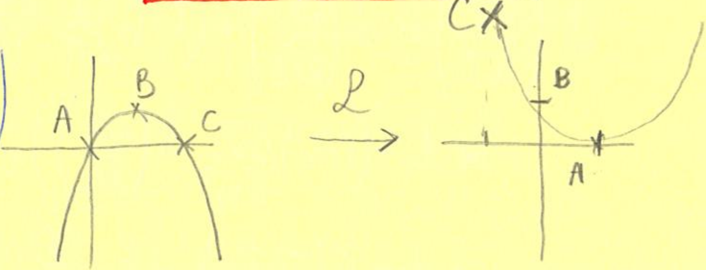
M1 General Physics ; Partial 2022

I) Basic questions

3°) θ is the in plane angle wrt a direction of reference (the director):

$$\boxed{S = 2\langle \cos^2\theta\rangle - 1}$$

5°)



6°) We take $f(x) = +x(2-x)$

$$p \equiv f'(x) = 2 - 2x \iff x = 1 - \frac{p}{2}$$

$$\hat{f}(p) = f(x) - xp \quad \text{with } x(p) = 1 - \frac{p}{2}$$

$$= \left(1-\frac{p}{2}\right)\left(1+\frac{p}{2}\right) - p\left(1-\frac{p}{2}\right)$$

$$= \left(1-\frac{p}{2}\right)\left[1+\frac{p}{2}-p\right]$$

$$\boxed{\hat{f}(p) = \left(1-\frac{p}{2}\right)^2}$$, compatible with the sketch.

II)

1°) First principle of thermodynamics

2°) $F = U - TS$; $\tilde{F} = U + MB - TS$

$\tilde{U} = U + MB$;

3°) $dU = TdS - MdB \Rightarrow \left.\frac{\partial T}{\partial B}\right|_M = -\left.\frac{\partial M}{\partial S}\right|_B$

$d\tilde{U} = TdS + BdM \Rightarrow \left.\frac{\partial T}{\partial M}\right|_S = \left.\frac{\partial B}{\partial S}\right|_M$

$dF = -SdT - MdB \Rightarrow \left.\frac{\partial S}{\partial B}\right|_T = \left.\frac{\partial M}{\partial T}\right|_B$

$d\tilde{F} = -SdT + BdM \Rightarrow \left.\frac{\partial S}{\partial M}\right|_T = -\left.\frac{\partial B}{\partial T}\right|_M$

4°) $c_B = T\left.\frac{\partial S}{\partial T}\right|_B$; $c_M = T\left.\frac{\partial S}{\partial T}\right|_M$

5°) $\chi = \left.\frac{\partial M}{\partial B}\right|_T$

6°) $\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial S}{\partial T}\right|_B + \left.\frac{\partial S}{\partial B}\right|_T \left.\frac{\partial B}{\partial T}\right|_M$

$$\Rightarrow \chi(c_B - c_M) = \left.\frac{\partial M}{\partial B}\right|_T \left(T\left.\frac{\partial S}{\partial T}\right|_B - T\left.\frac{\partial S}{\partial T}\right|_M\right)$$

$$= -\left.\frac{\partial M}{\partial B}\right|_T \left.\frac{\partial S}{\partial B}\right|_T \left.\frac{\partial B}{\partial T}\right|_M T$$

$$\left.\frac{\partial S}{\partial B}\right|_T = \left.\frac{\partial M}{\partial T}\right|_B \text{, see 3°}$$

$$\Rightarrow \boxed{\chi(c_B - c_M) = -T\left.\frac{\partial M}{\partial B}\right|_T \left.\frac{\partial M}{\partial T}\right|_B \left.\frac{\partial B}{\partial T}\right|_M}$$

7°) $\left.\frac{\partial M}{\partial B}\right|_T \left.\frac{\partial B}{\partial T}\right|_M \left.\frac{\partial T}{\partial M}\right|_B = -1$

$$\Rightarrow \left.\frac{\partial M}{\partial B}\right|_T \left.\frac{\partial B}{\partial T}\right|_M = -\left.\frac{\partial M}{\partial T}\right|_B$$

$$\Rightarrow \boxed{\chi(c_B - c_M) = T\left(\left.\frac{\partial M}{\partial T}\right|_B\right)^2}$$

8°) For $T < T_c$, $T \to T_c^-$

$c_B \propto (T_c - T)^{-\alpha}$

$M \propto (T_c - T)^{\beta}$

$\chi \propto (T_c - T)^{-\gamma}$

9°) Since $c_M > 0$, $\chi > 0$,

$$c_B \geqslant \underbrace{T\left(\left.\frac{\partial M}{\partial T}\right|_B\right)^2 \chi^{-1}}$$

$(T_c - T)^{-\alpha}$ $\qquad \propto T_c\left[(T_c-T)^{\beta-1}\right]^2 (T_c-T)^{\gamma}$

$$\Rightarrow \alpha \geqslant 2 - 2\beta - \gamma$$

$$\boxed{\alpha + 2\beta + \gamma \geqslant 2}$$; $a = 2$

10°) Ising 2d: $\alpha + 2\beta + \gamma = 0 + \frac{1}{4} + \frac{7}{4} = 2$

Thus, $\alpha + 2\beta + \gamma$ hits its lower bound.

Ising 3d $\alpha + 2\beta + \gamma \simeq 0{,}1 + 0{,}66 + 1{,}24 \simeq 2$

Again $\alpha + 2\beta + \gamma$ seems to hit the lower bound. The reason is that for all universality classes, we have

$$\alpha + 2\beta + \gamma = 2 \quad \text{(not shown)}$$

11°) A set of critical exponents is attached to a universality class; the latter changes when space dimension changes

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3) Onsager's exact solution

1°) J is the exchange / coupling constant

2°) At small T, $sh\left(\frac{2J}{kT}\right) \gg 1$ and thus $M \simeq 1$. When $T \nearrow$ $\frac{1}{sh^4\left(\frac{2J}{kT}\right)} \nearrow$ and $M \searrow$. We get

$M = 0$ for $\boxed{sh\left(\frac{2J}{kT_c}\right) = 1}$.

NB: $sh\left(\frac{2J}{kT_c}\right) = 1 \Rightarrow ch\left(\frac{2J}{kT_c}\right) = \sqrt{2}$

$$\Rightarrow th\left(\frac{2J}{kT_c}\right) = \frac{1}{\sqrt{2}}$$

$$argth\, x = \frac{1}{2}\log\frac{1+x}{1-x}$$

$$\Rightarrow argth\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log\left(\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}\right) = \frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = (\sqrt{2}+1)^2 \Rightarrow argth\left(\frac{1}{\sqrt{2}}\right) = \log(1+\sqrt{2})$$

$$\Rightarrow \boxed{kT_c = \frac{2J}{\log(1+\sqrt{2})}}$$

3°) We have information on $M(T)$ at $B = 0$. This gives access to one critical exp, β. Expand M^8 vs T in the vicinity of T_c

$$sh\left(\frac{2J}{kT}\right) \simeq sh\left(\frac{2J}{kT_c}\right) + (T - T_c)\frac{d}{dT} sh\left(\frac{2J}{kT}\right)\Big|_{T_c}$$

$$\simeq 1 - (T - T_c)\frac{2J}{kT_c^2}\left(ch\left(\frac{2J}{kT_c}\right)\right)\sqrt{2}$$

$$\simeq 1 + b(T_c - T), \quad b = \frac{2J}{kT_c^2}\sqrt{2} > 0$$

$$M^8 = 1 - \frac{1}{sh^4\left(\frac{2J}{kT}\right)}$$

$$\simeq 1 - \frac{1}{1 + b(T_c - T)}$$

$$\simeq 1 - [1 - b(T_c - T)]$$

$$\simeq b(T_c - T)$$

$M \propto (T_c - T)^{1/8}$; $\boxed{\beta = \frac{1}{8}}$