M1 GP Partial exam, February 2023

(A) 3°) $$\boxed{p(x) = \frac{1}{\sqrt{2\pi \times 1}} \exp\left[-\frac{(x-2)^2}{2}\right]}$$

$$\boxed{\langle X \rangle = 2}$$

$$\langle X^2 \rangle - \langle X \rangle^2 = 1 \Rightarrow \boxed{\langle X^2 \rangle = 5}$$

$$\langle (X - \langle X \rangle)^3 \rangle = 0 \text{ by symmetry}$$

$$\Rightarrow \langle X^3 - 3X^2\langle X \rangle + 3\langle X \rangle \langle X \rangle^2 - \langle X \rangle^3 \rangle = 0$$

$$\Rightarrow \langle X^3 \rangle = 3\langle X^2 \rangle \langle X \rangle - 3\langle X \rangle^3 + \langle X \rangle^3$$

$$= 3 \cdot 5 \cdot 2 - 3 \cdot 2^3 + 2^3$$

$$\boxed{\langle X^3 \rangle = 14}$$

Besides, it was seen in the tutorial that for a gaussian of mean m and standard deviation σ : $\langle e^{kX} \rangle = e^{km} e^{k^2\sigma^2/2}$

$$\Rightarrow \langle e^X \rangle = e^2 e^{1/2}$$

$$\boxed{\langle e^X \rangle = e^{5/2}}$$

(B) Transfer matrix formalism

1°) For the present model, $Z_N(\beta, J, h)$ and more precisely : $Z_N(\beta J, \beta h)$

2°) This is given by the number of configurations at the microscopic level : 3^N

3°) From the periodicity of the boundary conditions, we have

$$\sum_{i=1}^N S_i = \sum_{i=1}^N S_{i+1}$$

$$\Rightarrow H = -J \sum_{i=1}^N S_i S_{i+1} - \frac{h}{2} \sum_{i=1}^N (S_i + S_{i+1})$$

$$Z = \sum_{S_1, \dots S_N} \prod_{i=1}^N T(S_i, S_{i+1})$$

$$T(S, S') = e^{\overset{K}{\beta J} S S' + \overset{H}{\beta h}/2\, (S + S')}$$

$$T = \begin{array}{c} \begin{pmatrix} e^{K+H} & e^{H/2} & e^{-K} \\ e^{H/2} & 1 & e^{-H/2} \\ e^{-K} & e^{-H/2} & e^{K-H} \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ S' = 1 & S' = 0 & S' = -1 \end{array} \end{array}$$

$$\boxed{Z = \mathrm{Tr}\left[T^N\right]}$$

4°) Let us denote t_+ the largest eigenvalue of T, that is not degenerate, by virtue of Perron Frobenius theorem :

$$f = -kT \log t_+$$

5°) Here, $H = -J \sum_{i=1}^N \delta_{S_i, S_{i+1}}$ where the δ yields 1 if $S_i = S_{i+1}$, and 0 otherwise.

$$T = \begin{pmatrix} e^K & 1 & 1 \\ 1 & e^K & 1 \\ 1 & 1 & e^K \end{pmatrix}$$

with eigenvalues $e^K + 2$, and $e^K - 1$ (doubly degenerate).

$$\Rightarrow f = -kT \log(2 + e^K)$$