Should be written on a separate paper.

Electronic calculators, cell phones, and documents of all kinds are not allowed, except for lecture notes. Concise but explicative answers are expected throughout ; no bonus for verboseness. Write as neatly as possible and always specify clearly which question you are answering. No questions will be answered during the exam, for the sake of quietness and equity: if you detect what you believe to be an error or an inconsistency, explain so in your answer, and move on; you are also judged on your ability to understand the questions raised.

A Exercises

1) Take a Gaussian random variable X with mean 1 and standard deviation 2. Write the probability density function, then compute $\langle X \rangle$, $\langle X^2 \rangle$, $\langle X^3 \rangle$ and $\langle e^X \rangle$.

2) We consider a 3-state spin model on a line, i.e. where a given spin can take three distinct values. The Hamiltonian is such that like spins (ie spins of the same value) interact with energy $-J$, while unlike spin interactions yield a $+J$ energy. Only nearest neighbor spins do interact. Write the transfer matrix, when no external field is applied. Same question for a 4-spin model of the same kind.

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B Landau theory and tricriticality

Within a Landau approach, the free energy is expanded in powers of the order parameter ϕ, in the form

$$\mathcal{F} = \frac{1}{2} a_2 \phi^2 + \frac{1}{4} a_4 \phi^4 + \frac{1}{6} a_6 \phi^6 , \tag{1}$$

where in the vicinity of T_c, the coefficient a_2 linearly depends on temperature T: $a_2 = (T - T_c)\,\widetilde{a}_2$ (with $\widetilde{a}_2 > 0$). For simplicity, we suppose that a_4 and a_6 are independent on temperature.

B.1 Order of transitions and sign of coefficients

1. Provide an example of a physical system where such an expansion may be relevant.
2. What should the sign of a_6 be?
3. Sketch the free energy profiles $\mathcal{F}(\phi)$ for different temperatures, treating separately the cases $a_4 > 0$ and $a_4 < 0$. For each case, what is the order of the corresponding phase transition embodied in Eq. (1)?
4. In the first order transition case, we denote T^* the critical temperature. What are the conditions on $\mathcal{F}$ which determine this temperature? At $T = T^*$, give the values of ϕ that minimize the free energy and show that

   $$T^* = T_c + \alpha \frac{a_4^2}{\widetilde{a}_2\, a_6} .$$

   What is the value of α?
5. Draw a schematic phase diagram in the plane (T, a_4), for fixed values of $\widetilde{a}_2$ and a_6. Indicate the phase transition lines of first and second orders. In which point of the diagram do they meet?

B.2 Study of the tricritical point

We will now assume that $a_4 = 0$, which defines for our model a so-called tricritical point.

1. What is the order of the phase transition ?

2. How does the order parameter depend on temperature, in the vicinity of T_c? Infer from this behaviour the value of the β exponent, defined by $\phi \propto (T_c - T)^\beta$ below the critical point. What is the value of β when $a_4 > 0$?

3. Under the action of an applied external field B, which additional term should appear in the free energy ? What is the exponent δ which measures, at $T = T_c$, the response to B through $\phi \propto B^{1/\delta}$? Same question when $a_4 > 0$.