M1 GP - Statistical Physics - Exam April 2023

A 1°)
$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]}$$

with $m = 1$ and $\sigma = 2$.

$\boxed{\langle X \rangle = 1}$ by definition

$\langle X^2 \rangle - \langle X \rangle^2 = \sigma^2 = 4$ by definition

$\Rightarrow \boxed{\langle X^2 \rangle = 5}$

$\langle (X - \langle X \rangle)^3 \rangle = 0$ by symmetry

$\Rightarrow \langle X^3 - 3X^2\langle X \rangle + 3X\langle X \rangle^2 - \langle X \rangle^3 \rangle = 0$

$$\langle X^3 \rangle = 3\langle X^2 \rangle \langle X \rangle - 2\langle X \rangle^3$$

$$= 3 \cdot 5 - 2$$

$$\boxed{\langle X^3 \rangle = 13}$$

$$\langle e^{kX} \rangle = e^{km} e^{k^2\sigma^2/2}$$

$$\Rightarrow \boxed{\langle e^X \rangle = e\, e^2 = e^3}$$

2°)
$$\mathbb{T} = \begin{pmatrix} e^{+\beta J} & e^{-\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{-\beta J} & e^{\beta J} \end{pmatrix}$$

In the case of a 4-state spin:

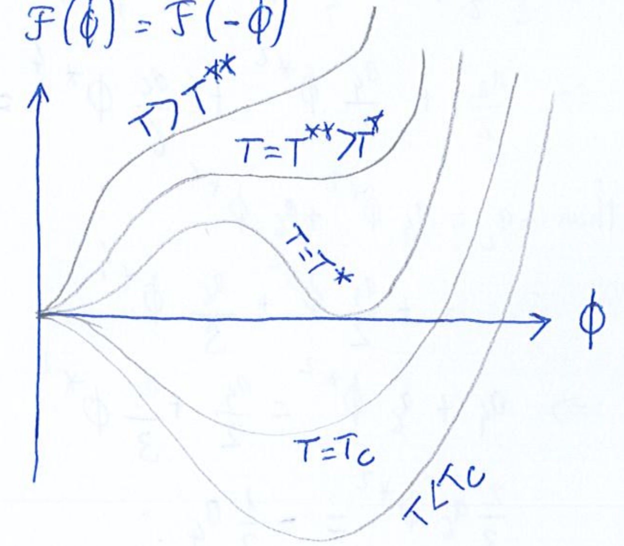
$$\mathbb{T} = \begin{pmatrix} e^{\beta J} & e^{-\beta J} & e^{-\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} & e^{-\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{-\beta J} & e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{-\beta J} & e^{-\beta J} & e^{\beta J} \end{pmatrix}$$

B1 1°) The standard ferromagnetic Ising model with spins 1/2.

2°) $\boxed{a_6 > 0}$ to garantee stability

3°) $\mathcal{F}(\phi) = \mathcal{F}(-\phi)$

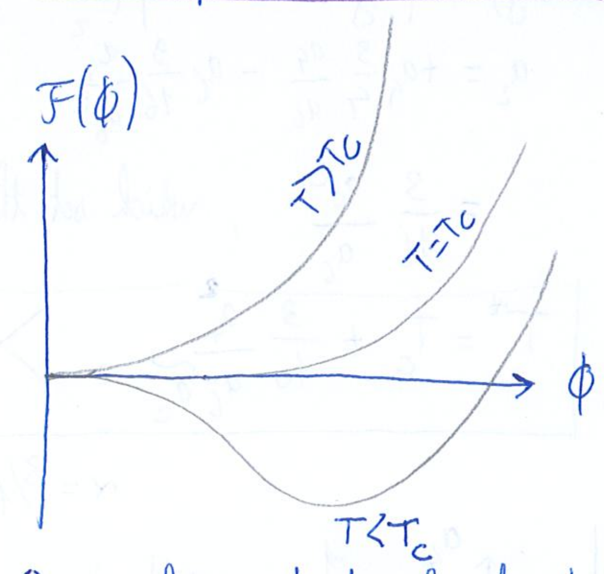
$a_4 < 0$



$T_c < T^* < T^{**}$

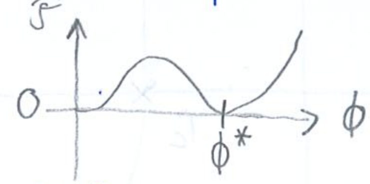
First order phase transition at $T = T^*$

$a_4 > 0$



Scenario of a second order phase transition

4°) For $T = T^*$:



$\mathcal{F}(0) = 0 = \mathcal{F}(\phi^*)$

$\mathcal{F}'(\phi^*) = 0$ and $\phi^* \neq 0$

$$F'(\phi^*) = 0 = a_2 \phi^* + a_4 \phi^{*3} + a_6 \phi^{*5}$$

and since $\phi^* \neq 0$, we get

$$a_2 + a_4 \phi^{*2} + a_6 \phi^{*4} = 0$$

with $F(\phi^*) = 0$

$$\Rightarrow \frac{1}{2} a_2 \phi^{*2} + \frac{1}{4} a_4 \phi^{*4} + \frac{1}{6} a_6 \phi^{*6} = 0$$

$$\Rightarrow \frac{a_2}{2} + \frac{a_4}{4} \phi^{*2} + \frac{a_6}{6} \phi^{*4} = 0$$

Thus $-a_2 = a_4 \phi^{*2} + a_6 \phi^{*4}$ (1)

$$= + \frac{a_4}{2} \phi^{*2} + \frac{a_6}{3} \phi^{*4}$$

$$\Rightarrow a_4 + a_6 \phi^{*2} = \frac{a_4}{2} + \frac{a_6}{3} \phi^{*2}$$

$$\frac{2}{3} a_6 \phi^{*2} = -\frac{1}{2} a_4$$

$$\phi^{*2} = -\frac{3}{4} \frac{a_4}{a_6} > 0$$

Finally, we plug back in Eq (1) to find

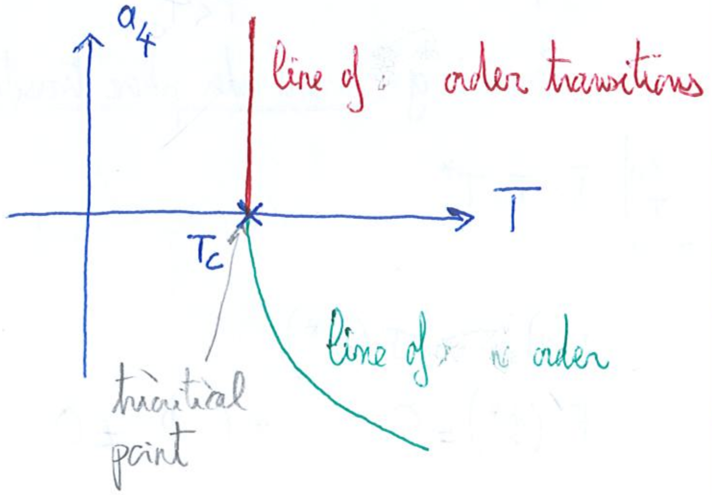
$$a_2 = +a_4 \frac{3}{4} \frac{a_4}{a_6} - a_6 \frac{9}{16} \frac{a_4^2}{a_6^2}$$

$$= \frac{3}{16} \frac{a_4^2}{a_6}$$, which set the temper:

$$\boxed{T^* = T_c + \frac{3}{16} \frac{a_4^2}{a_6 \tilde{a}_2} > T_c}$$

$\alpha = 3/16$

5°)



B2 1°) For $a_4 = 0$, the scenario is of the same type as with $a_4 > 0$, hence 2nd order.

2°) The order parameter is set by $F'(\phi) = 0$, while $\phi \neq 0$

$$\Rightarrow a_2 \phi + a_6 \phi^5 = 0$$

$$\Rightarrow a_2 + a_6 \phi^4 = 0$$

$$\Rightarrow \phi = \left(-\frac{a_2}{a_6}\right)^{1/4} = \left[\frac{\tilde{a}_2 (T_c - T)}{a_6}\right]^{1/4}$$

and thus $\boxed{\beta = 1/4}$. $\boxed{\text{For } a_4 > 0, \ \beta = 1/2}$

3°) For $B \neq 0$, the magnetic-like free energy reads

$$F(\phi) = \frac{a_2}{2} \phi^2 + \frac{a_4}{4} \phi^4 + \frac{a_6}{6} \phi^6 - B\phi$$

At $T = T_c$, $a_2 = 0$ and here, $a_4 = 0$.

Thus, the equation of state follows from

$$\frac{\partial F}{\partial \phi} = 0$$

$$= a_6 \phi^5 - B$$

$$\Rightarrow \boxed{\delta = 5}$$

For $a_4 > 0$, we get at $T = T_c$:

$$a_4 \phi^3 + a_6 \phi^5 = B$$

meaning that $\boxed{\delta = 3}$, ie the same (mean-field) result as when $a_6 = 0$.