Electronic calculators, cell phones, and documents of all kinds are not allowed. Concise but explicative answers are expected throughout ; no bonus for verboseness. Write as neatly as possible and always specify clearly which question you are answering. No questions will be answered during the exam, for the sake of quietness and equity: if you detect what you believe to be an error or an inconsistency, explain so in your answer, and move on; you are also judged on your ability to understand the questions raised. The different sections are independent.

A Sanov, Gärtner-Ellis, Stirling and large deviations

Our interest goes to the sum of independent and identically distributed exponential random variables:

$$S_n = \sum_{i=1}^{n} x_i \qquad \text{where the } x_i \text{ have distribution } p_0(x) = \frac{1}{\mu} e^{-x/\mu}, \quad x \geq 0 \text{ and } \mu > 0. \tag{1}$$

1) Compute $\langle x_1 \rangle$, $\langle S_n \rangle$ together with the variances $V(x_1)$ and $V(S_n)$.

2) What information on the distribution of S_n, $P(S_n)$, does the central limit theorem yield? Write the corresponding $P(S_n)$. In which S_n region is it supposed to apply?

3) From Sanov theorem, compute the large deviation function of S_n, $\phi(z)$, where $z \equiv S_n/n$. Sketch schematically $\phi(z)$. For which value of z is it minimum ?

4) How can one retrieve the central limit result from the knowledge of the large deviation function? Show explicitly the connexion.

5) What is the extent of the "central limit region" (how does it depend on n?), and what is this region?

6) Gärtner-Ellis theorem (GE) offers an alternative derivation of the large deviation function. Remembering that GE relates the scaled cumulant generating function

$$\kappa_n(t) = \frac{1}{n} \log \left\langle e^{tnz} \right\rangle \tag{2}$$

to the Legendre transform of ϕ, through a saddle-point argument, compute first $\kappa_n(t)$, and then $\phi(z)$ (that should coincide with the result found in question 3). Why does κ_n not depend on n?

7) An explicit calculation of the pdf of S_n, $P(S_n)$ is possible, making use of the relation between S_n and Poisson distribution of rate μ. One finds a so-called Γ distribution:

$$P(S_n) = \frac{1}{\mu} e^{-S_n/\mu} \frac{(S_n/\mu)^{n-1}}{(n-1)!}. \tag{3}$$

a) From this, compute the large deviation function of S_n.

b) Propose a simple argument, combining relation (3) with the central limit theorem, that leads to Stirling formula for $n!$, at asymptotically large n.

$\longrightarrow$ Bonus question: show relation (3).

B Langevin, Itō-Doblin and Stratonovich

The velocity of a Brownian object obeys the Langevin equation

$$\dot{v} = -\gamma v + \sqrt{2\Gamma}\xi(t) \quad \text{with} \quad \langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'). \tag{4}$$

1) Use Itō-Doblin calculus to obtain the differential equation for $\dfrac{d}{dt} \langle v^2 \rangle$.

2) Solve this equation, assuming that $v = v_0$ at $t = 0$.

3) Same question with Stratonovich calculus.

4) How does the skewness $\left\langle (v - \langle v \rangle)^3 \right\rangle$ evolve in time?

C From geometric Brownian motion to sedimentation equilibrium

The model of geometric (or exponential) Brownian motion, used in the financial industry, provides an example of a Markov process with multiplicative noise. There, the stochastic process $S(t)$ obeys the Langevin equation

$$\frac{dS(t)}{dt} = \left(\mu - \frac{\sigma^2}{2}\right) S + \sigma\, S(t)\, \xi(t) \tag{5}$$

where $\xi(t)$ denotes a white noise with

$$\langle \xi(t) \rangle = 0 \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = \delta(t-t'). \tag{6}$$

Equation (5) is understood in the Stratonovich sense. Related frameworks appear in population genetics models, or in the study of birth-death processes. We will assume that $S(0) = 1$ (without loss of generality) and more importantly, that $\mu < \sigma^2/2$.

1) Show that the solution to Eq. (5) reads

$$S(t) = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + X(t)\right], \tag{7}$$

where $X(t)$ is a simple stochastic process. Specify completely this latter process.

2) Making use of relation (7) (*i.e* without computing the pdf of $S(t)$, but rather invoking that of $X(t)$), compute the moments of $S(t)$, $\langle S^n(t) \rangle$. Show in particular that

$$\langle S(t) \rangle = e^{\alpha \mu t}. \tag{8}$$

What is α?

3) From the probability distribution function of X at a given time, deduce that of S, $p(s,t)$. Show that in some intermediate s-window, it is of power-law type.

4) Write the Fokker-Planck equation obeyed by $p(s,t)$. Put it in the form

$$\frac{\partial p(s,t)}{\partial t} = (\ldots) + \frac{\sigma^2}{2} \frac{\partial}{\partial s} \left\{ s \frac{\partial [s\, p(s,t)]}{\partial s} \right\}. \tag{9}$$

Of course, specify what $(\ldots)$ refers to, in the above equation.

5) As specified, the problem does not admit a stationary solution. To obtain one, we need to impose an additional constraint, and we take a reflecting "wall" at S_{min}, so that $S \geq S_{\text{min}}$. What is the corresponding equilibrium solution to the Fokker-Planck equation (9)? Show that it is of power-law type. What is the corresponding Lévy index?

6) Working with the variable $z = \log S$, show that with the above condition $z \geq z_{\text{min}} \equiv \log S_{\text{min}}$, we have the same Langevin equation as for colloids in a gravitational field. What is then the equilibrium distribution of z? Recover the equilibrium Lévy distribution of S.

7) We go back to Eq. (5) that we write as

$$\frac{dS(t)}{dt} = \mu' S + \sigma\, S(t)\, \xi(t) \tag{10}$$

which we now interpret in the Itō sense, with an unspecified drift coefficient μ'. Write the corresponding Fokker-Planck equation. How should μ' be chosen in order to generate the same dynamics as the Stratonovich formulation?

D Martingales, first passage properties and Feynman-Kac relation

We consider the Wiener process $\dot{X}(t) = \sqrt{2D}\, \xi(t)$ with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. At $t = 0$, we have $X(0) = 0$. We are interested in the first-passage time at $X = \pm a$, that we denote τ. This random variable is thus the exit time from the interval $[-a, a]$ (we take $a > 0$).

D.1 The martingale corner

1) Show that $X^2(t) - 2Dt$ is a martingale.

2) Invoking Doob stopping time theorem, compute the mean exit time $\langle \tau \rangle$.

3) We set out to construct more martingales, to infer information dealing with τ. Is it possible to have a martingale of the form $X^4(t) - \varphi(t)$, where $\varphi(t)$ is a suitably chosen function of time?

4) For which choice of $\psi(t)$ is $X^4(t) - 12DtX^2(t) + \psi(t)$ a martingale? We will take $\psi(0) = 0$.

5) What information on τ can we infer from the previous result?

6) A new type of martingale can be obtained from computing the exponential average of X. Show that $e^{\theta X(t) - \Phi(t)}$, for an arbitrary θ, is a martingale for a well-chosen $\Phi(t)$ (we will take $\Phi(0) = 0$).

7) From the previous result and the stopping-time theorem, show that taking $\theta = \sqrt{n}/a$ yields the moment generating function in the form

$$\left\langle e^{-nD\tau/a^2} \right\rangle = \frac{1}{\Gamma(\sqrt{n})}. \tag{11}$$

What is the function Γ?

8) From this, compute $\langle \tau \rangle$ and $\langle \tau^2 \rangle$, and check that you recover previously obtained results.

D.2 The Feynman-Kac confirmation

We wish to check the results conveyed by Eq. (11), from the Feynman-Kac relation. For the Brownian motion under study here, we define the first-passage functional

$$Q(x_0) = \left\langle e^{-\int_0^\tau V[x(t')]dt'} \right\rangle, \tag{12}$$

where the brackets stand for an average over all trajectories emerging from the point $x(0) = x_0$, τ is again the first passage time at $X = \pm a$ and V is so far an unspecified function.

1) What is the differential equation obeyed by $Q(x_0)$?

2) What are the associated boundary conditions?

3) How should we choose V to get the moment generating function as in (11)? Do so, and compute explicitly $Q(x_0)$. Conclude, after having taken the appropriate value of x_0.

D.3 The martingale factory

We have considered above a number of martingales, that are all of the form $f(X(t), t)$. We are interested on the conditions on $f(x, t)$ such that the martingale property holds. For all times $t > s$, we demand that the conditional average

$$\langle f(X(t), t) | X(s) \rangle = f(X(s), s). \tag{13}$$

1) Write the forward Fokker-Planck equation obeyed by the conditional density $p(x, t | X(s), s)$.

2) How is $\langle f(X(t), t) | X(s) \rangle$ related to $p(x, t | X(s), s)$?

3) Show that it is sufficient that f obeys a diffusion-like equation (which one?) to fulfill Eq. (13). Check that all previously obtained martingales obey this condition.