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1 From the central limit theorem to Stirling's formula. . . and back

1) We take here advantage of the stability of Poisson random variables upon addition, to deduce Stirling's approximation for $n!$ when n is large, from the central limit theorem.

a) We denote $\mathcal{P}(\lambda)$ the law of a Poisson random variable with parameter λ. What is the law for the sum of two independent such variables? In the remainder, we take $\lambda = 1$ for simplicity.

b) We define $S_n = \sum_{i=1}^{n} x_i$ where the x_i are IID Poisson variables $\mathcal{P}(\lambda = 1)$. How does the central limit theorem constrain the distribution of S_n for n large?

c) By computing the probability that $n - 1/2 < S_n < n + 1/2$ in two ways, show that when $n \to \infty$

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \tag{1}$$

2) We consider the simple random walk starting from the origin, where at each discrete time $n = 1, 2 \ldots$, a jump ± 1 is made, with equal probability $1/2$. Take n even.

a) What is the probability to return to the origin after n steps?

b) Use Stirling's formula to show that this probability behaves for large n as $\dfrac{\sqrt{2}}{\sqrt{\pi n}}$

c) What is the probability that the walker is at position m after n steps, with $m = 0, \pm 2, \pm 4 \ll n$?

d) Recover the central limit theorem result for the above probability. Beware "the factor 2".

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2 Consistency of Itō-Doblin and Stratonovich calculus

For a Langevin equation with additive noise, such as

$$\frac{dx}{dt} = \mu F(x) + \sqrt{2D}\,\eta(t) \qquad \text{with} \qquad \langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t+\tau) \rangle = \delta(\tau), \tag{2}$$

and an arbitrary function $\varphi(x)$, check the consistency of Itō-Doblin and Stratonovich calculus for computing

$$\left\langle \frac{d\varphi(x(t))}{dt} \right\rangle. \tag{3}$$

⋄

3 Itō-Doblin, Stratonovich and Wick

We consider the Wiener process $x(t)$ with diffusion coefficient D. It this process Gaussian? Invoking Wick theorem, compute the explicit time dependence of $\langle x^4 \rangle$ and $\langle x^6 \rangle$. Recover this results from both Itō-Doblin and Stratonovich routes, computing

$$\left\langle \frac{dx^4}{dt} \right\rangle \quad \text{and} \quad \left\langle \frac{dx^6}{dt} \right\rangle. \tag{4}$$

⋄

4 The Wiener process from the scaling limit of a random walk

We consider the symmetric random walk on $\mathbb{Z}$, with probabilities 1/2 to jump left or right, at each step. Starting from the origin, the walker has position x_n after n steps. From x_n, we define by linear interpolation the continuous time process $x_c(\widetilde{n})$, where $\widetilde{n}$ is now real, and $x_c(\widetilde{n}) = x_n$ when $\widetilde{n} = n$ is an integer. From this, we introduce the process

$$x_\varepsilon(t) = \varepsilon\, x_c\left(\frac{t}{\Delta t}\right). \tag{5}$$

What is the appropriate scaling limit, where $\varepsilon \to 0$, $\Delta t \to 0$, such that $x_\varepsilon(t) \to W(t)$, the Wiener process?

⋄

5 Markovian or non Markovian?

Our interest goes to a persistent random walk in 1D, where the probability to make a step in the same direction as the previous one is $p > 1/2$. With probability $1 - p$, the step is made in the reverse direction. Does the position of the walker after n steps, x_n, define a Markov process? How can define a Markov process here?

⋄

6 Position and velocity processes in the Langevin equation

We start with the 1D Langevin equation:

$$m\frac{d^2x}{dt^2} = -m\gamma\frac{dx}{dt} + R(t), \tag{6}$$

where $R(t)$ is a Gaussian process with $\langle R(t) \rangle = 0$ and $\langle R(t_1)R(t_2) \rangle = \Gamma m^2 \delta(t_1 - t_2)$. The goal is to study here the position and velocity processes that ensue, $x(t)$ and $v(t)$. Two types of initial conditions will be addressed:
(i) at $t = 0$, $x(0) = 0$ and $v(0) = v_0$ (fixed initial velocity);
(ii) at $t = 0$, $x(0) = 0$ and $v(0)$ are distributed according to the equilibrium Maxwellian.

1) Velocity process.

a) Express $v(t)$ as a function of $R(t)$, for a given realization of $R(t)$.

b) Characterize explicitly the process, ie write the n-times probability densities for all n, for each of the initial conditions (i) and (ii). Is $v(t)$ Markovian? Gaussian? Stationary?

2) Position process.

a) Express $x(t)$ as a function of $v(t)$.

b) Characterize explicitly the process $x(t)$, for each of the initial conditions (i) and (ii). Is $x(t)$ Markovian? Gaussian? Stationary?

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7 Doob's theorem (1942)

The goal here is to show that ***a Gaussian stationnary process is Markovian iff its autocorrelation function is exponential.***

1) We consider first a Gaussian, stationnary and Markovian process. For the sake of simplicity, we take the mean value to vanish and the variance to be unity, without loss of generality.

a) Write explicitly the one-time probability distribution function $p(x)$.

b) Write the transition probability $p(y, \tau|x, 0)$. It involves four parameters that we set out to determine in the following questions.

c) Express the normalization condition of the transition probability.

d) Write the consistency condition fulfilled by $p(x)$ and $p(y, \tau|x, 0)$.

e) Show then that

$$p(y, \tau|x, 0) = \frac{1}{\sqrt{2\pi(1-\gamma^2)}} e^{-\frac{(y-\gamma x)^2}{2(1-\gamma^2)}}, \tag{7}$$

where γ is some parameter.

f) Using previous results, compute the correlation function $\langle x(0)x(\tau)\rangle$ as a function of γ. What tis the physical sugnificance of γ?

g) Write Chapman-Kolmogorov equation for the process, and show that the correlation function is exponential.

2) Show the reciprocal statement: a process that is Gaussian, stationnary, and with an exponential correlation function, is Markovian.

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8 Fokker-Planck from Itō-Doblin calculus

We consider the Langevin equation $\dot{x} = \mu F(x) + \sqrt{2D}\eta(t)$ where $\eta(t)$ is a Gaussian white noise with correlation $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. Take an arbitrary test function $\varphi(x)$ and compute $d\langle \varphi(x)\rangle/dt$. Using the fact that

$$\langle \varphi(x(t))\rangle = \int dx\, P(x,t)\,\varphi(x) \tag{8}$$

where $P(x,t)$ is the position p.d.f at time t, derive the Fokker-Planck equation for $P(x,t)$.

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9 Several solutions to the diffusion equation

We aim here at solving the diffusion equation

$$\frac{\partial P}{\partial t} = D\,\frac{\partial^2 P}{\partial x^2}, \qquad t \geq t_0, \qquad \text{with the initial condition} \quad P(x, t_0|x_0, t_0) = \delta(x - x_0). \tag{9}$$

We thus work with the conditional density $P(x, t|x_0, t_0)$.

1) A first approach is to search for a scaling solution, with a form that would exhibit diffusive scaling. This amounts to choosing

$$P(x, t|x_0, t_0) = \psi(t)\,\varphi\left(\frac{x - x_0}{\sqrt{t - t_0}}\right). \tag{10}$$

Show that normalization of P specifies the time dependence of $\psi(t)$. Injecting the above form into (9), otain and solve the ordinary differential equation fulfilled by φ, to get finally

$$P(x, t|x_0, t_0) = \frac{1}{\sqrt{4\pi D(t - t_0)}}\,\exp\left[-\frac{(x - x_0)^2}{4D(t - t_0)}\right]. \tag{11}$$

2) A second method consists in taking the Laplace transform of Eq. (9). With $\tau = t - t_0$, we introduce

$$\widetilde{P}(x, s) = \mathcal{L}_s\left[P(x, \tau)\right] = \int_0^\infty P(x, \tau)\, e^{-s\tau}\, d\tau. \tag{12}$$

Solve for $\widetilde{P}(x, s)$. Using the inverse Laplace transform

$$\mathcal{L}_s^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \frac{e^{-a^2/(4\tau)}}{\sqrt{\pi\tau}}, \tag{13}$$

recover Eq. (11). For completeness, show relation (13), e.g. with a complex analysis approach.

3) A third method bears some similarities with quantum mechanics. We consider the operator $\widehat{H} \equiv -D\,\partial^2/\partial x^2$. Assume first that the spectrum of $\widehat{H}$ is discrete, with a set of eigenvalues λ and associated eigenfunctions $\psi_\lambda(x)$. We assume that the latter form a complete set:

$$\sum_\lambda \psi_\lambda(x)\,\psi_\lambda^*(x_0) = \delta(x - x_0), \tag{14}$$

where the star denotes complex conjugate. Write the orthogonality condition between the eigenfunctions. Show that

$$P(x, t|x_0, t_0) = \sum_\lambda e^{-\lambda(t - t_0)}\,\psi_\lambda(x)\,\psi_\lambda^*(x_0). \tag{15}$$

Show that in the present case though, the spectrum is continuous, indexed by wavenumber k such that $\lambda = Dk^2$, with (normalized) eigenfunctions

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}}\,e^{ikx}. \tag{16}$$

Rewriting Eq. (15) in the present situation as

$$P(x, t|x_0, t_0) = \int_{-\infty}^{\infty} e^{-Dk^2 t} e^{ik(x - x_0)}\,\frac{dk}{2\pi}, \tag{17}$$

recover the solution (11). This method leads here to a Fourier Transform approach (check this).

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10 Ornstein-Uhlenbeck, linear response, fluctuation-dissipation theorem

The Ornstein-Uhlenbeck process describes the velocity relaxation of a Brownian object subject to viscous friction, in the absence of an external force. Alternatively, the same framework also applies to an overdamped dynamics in a harmonic potential

$$U(x) = \frac{\kappa}{2} x^2. \tag{18}$$

We will assume that a small x-independent force is applied to the "particle", so that

$$\frac{dx}{dt} = -\mu\kappa x + \mu f(t) + \sqrt{2D}\,\xi(t), \tag{19}$$

with $\xi(t)$ a Gaussian white noise:

$$\langle \xi(t) \rangle = 0 \qquad ; \qquad \langle \xi(t)\xi(t') \rangle = \delta(t-t'). \tag{20}$$

1) What is the physical meaning of D and μ? How are they related at equilibrium? Equilibrium refers here to the suspending fluid in which the particle is immersed.

2) Solve Eq. (19) for the trajectory $x(t)$, with initial conditions $x = x_0$ at $t = t_0$. Compute $\langle x(t) \rangle$ for $f(t) \neq 0$.

3) Express $\langle x(t)x(t') \rangle_{\text{eq}}$ for $f = 0$, *i.e.* at equilibrium. Check that the results only depends on $t - t'$.

On general grounds, the fluctuation-dissipation theorem relates a correlation function at equilibrium to the response function associated to a small externally applied force. Here, we are interested in the response of $x(t)$ to $f(t)$, and the response function (which actually is an xx response function) is defined as

$$\langle x(t) \rangle = \int_{-\infty}^{t} \chi(t-t')\, f(t')\, dt'. \tag{21}$$

The fluctuation-dissipation theorem states (with $\beta^{-1} = kT$ and $\theta(t)$ the Heaviside function) that

$$\boxed{\chi(\tau) = -\beta\, \theta(t)\, \frac{d}{d\tau} \langle x(\tau)x(0) \rangle_{\text{eq}}}. \tag{22}$$

4) Before checking explicitly the above relation between correlation and response functions, we consider a static force $f(t) = f_0$, to realize that Eq. (22) hides the equipartition theorem. First, what is $\langle x \rangle$ in this case? No calculation is necessary, just a sensical remark on the behavior of springs under constant tension. Check that you recover this expectation from the results of question 2 (take $t_0 \to -\infty$ since we look at the static response). Second, making use of Eqs. (21) and (22), together with an integration by parts, show that $\langle x^2 \rangle_{\text{eq}} = kT/\kappa$.

5) Compute $\chi(\tau)$ for an equilibrium system, that thus has had enough time to relax. Check that the fluctuation-dissipation theorem (22) is obeyed.

6) Discuss the Brownian limit $\kappa \to 0$.

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11 Two generalized Feynman-Kac relations

We consider the overdamped Langevin process

$$\frac{dx(t)}{dt} = \mu F(x,t) + \sqrt{2D}\,\xi(t) \tag{23}$$

where $\xi(t)$ is a Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$.

1) Two arbitrary functions $V(x,t)$ and $f(x,t)$ being chosen, we define the functional of the trajectory:

$$Q(x_0,t,t_0) = \left\langle f(x(t),t)\, e^{-\int_{t_0}^{t} V[x(\tau),\tau]d\tau} \right\rangle \tag{24}$$

where $t_0 \le t$ and the average over trajectories, represented by the bracket above, is performed for a fixed initial condition $x(t_0) = x_0$. Show that

$$\frac{\partial Q}{\partial t_0} + D\frac{\partial^2 Q}{\partial x_0^2} + \mu F(x_0,t_0)\,\frac{\partial Q}{\partial x_0} - V(x_0,t_0)\,Q = 0 \tag{25}$$

with the condition $Q(x_0,t_0,t_0) = f(x_0,t_0)$ (sometimes called the terminal condition, since it corresponds to $t_0 = t$). To this end, one may introduce the auxiliary propagator

$$p(x,\Omega,t|x_0,\Omega_0,t_0) \qquad \text{where} \qquad \Omega(t,t_0) = \Omega_0 + \int_{t_0}^{t} V[x(\tau),\tau]\,d\tau, \tag{26}$$

obeying a backwards Fokker-Planck equation.

In the case where $V = 0$, what evolution equation do we get?

2) We consider a variant, with the functional

$$\mathcal{Q}(x_0,t,t_0) = \int_{t_0}^{t} dt' \left\langle f(x(t'),t')\, e^{-\int_{t_0}^{t'} V[x(\tau),\tau]d\tau} \right\rangle, \tag{27}$$

where the bracket is again an average over trajectories starting from $x(t_0) = x_0$. Show that

$$\frac{\partial \mathcal{Q}}{\partial t_0} + D\frac{\partial^2 \mathcal{Q}}{\partial x_0^2} + \mu F(x_0,t_0)\,\frac{\partial \mathcal{Q}}{\partial x_0} - V(x_0,t_0)\,\mathcal{Q} + f(x_0,t_0) = 0. \tag{28}$$

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12 Martingales for the asymmetric random walk

We are interested in the asymmetric random walk on $\mathbb{Z}$, with probabilities p and $q = 1-p$ to jump right and left respectively. The walker starts at $X_0 = 0$ and its position after n steps is denoted X_n. Show that

$$M_n \equiv \left(\frac{q}{p}\right)^{X_n} \qquad \text{and} \qquad M'_n \equiv X_n - n(p-p) \tag{29}$$

are martingales. Knowing this, use Doob's stopping time theorem to compute the probability that the walker leaves a predefined interval $[-a,b]$ by the right boundary at b (we take $a > 0$). Same question with the left boundary at $-a$. What is finally the mean exit time from the interval? Check that you recover previously known results.

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13 Algebraic area enclosed by a random walk

Among all possible symmetric random walks on the square lattice ($\mathbb{Z}^2$), we restrict to those forming closed paths (i.e. ending at their starting point, see Fig. 1).

1) Show that there are $\binom{n}{n/2}^2$ such walks. To this end, one can partition these walks according to k, the number of steps made along the x axis. There are conversely $n-k$ steps made along the y axis. Note that n and k should be even. It may be useful here to use the relation

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p}{n-j} = \sum_{j=0}^{p} \binom{p}{j}^2 = \binom{2p}{p} \tag{30}$$

which stems from counting in two ways the number of teams of p players one can form out of a group of $2p$ players. Among the $2p$ players, p have a cap, and p do not. Thus, in the team formed, there can be $j = 0, 1 \ldots$ up to $j = p$ players with a cap.

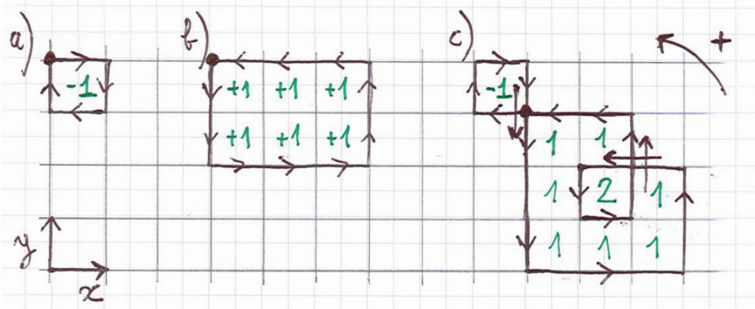

Figure 1: A few closed random walks on $\mathbb{Z}^2$ with: a) $n = 4$ steps and enclosed area $\mathcal{A} = -1$; b) $n = 10$ steps and $\mathcal{A} = 6$; c) $n = 20$ steps and $\mathcal{A} = -1 + 7 \times 1 + 2 = 8$. In panel c), the unit cell indicated with a 2 is enclosed twice by the walk and contributes a $+2$ to $\mathcal{A}$; 7 cells are enclosed once and clockwise, 1 cell is enclosed once counterclockwise.

2) For each closed walk, we define the algebraic area $\mathcal{A}$ as the total oriented area spanned by the walk, as it traces the lattice. A unit lattice cell enclosed counterclockwise (resp. clockwise) counts $+1$ (resp. -1). Besides, if the cell is enclosed more than once, its area is counted with its multiplicity, see Fig 1. We denote $C_n(\mathcal{A})$ the number of walks of n steps that have algebraic area $\mathcal{A}$ ($\mathcal{A} = 0, \pm 1, \pm 2 \ldots$ and $C_n(\mathcal{A}) = C_n(-\mathcal{A})$ by symmetry). The trick is to introduce two hopping operators, r (one step to the right, along x) and u (one step up, along y), which provides a natural way to represent a given walk: $ru^{-1}r^{-1}u$ for the path in Fig. 1-a). The operators do not commute and it is convenient to declare $ru = Qur$, where $Q \neq 1$ is some arbitrary number. With this rule, check that a closed walk, defined by a sequence of r, r^{-1}, u and u^{-1} yields $Q^{\mathcal{A}}$. From this remark, it follows that when selecting the u and r independent part in $(r + r^{-1} + u + u^{-1})^n$, we obtain $C_n(\mathcal{A})$:

$$(r + r^{-1} + u + u^{-1})^n = \sum_{\mathcal{A}=-\infty}^{\infty} C_n(\mathcal{A})\, Q^{\mathcal{A}} + \ldots \tag{31}$$

where $\ldots$ is for terms that explicitly depend on the operators (such as r^n, $r^{n-1}u$ etc.) and are associated to non-closed walks. For instance, $(r + r^{-1} + u + u^{-1})^4 = 28 + 4Q + 4Q^{-1} + \ldots$ meaning that among the $\binom{4}{2}^2 = 36$ closed walks of 4 steps, $C_4(0) = 28$ enclose a vanishing area, $C_4(1) = 4$ enclose an area $+1$ and $C_4(-1) = 4$ enclose an area -1 (you may check explicitly the latter two results).