$\boxed{\text{Gaussian calculus}}$ 1

$X \rightsquigarrow \mathcal{G}(m,\sigma)$; $\langle e^{kX}\rangle = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{kx}\, e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$; $y = x-m$

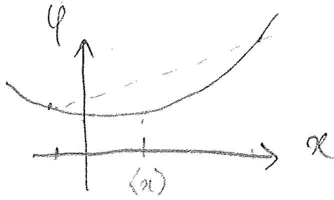
$$\langle e^{kX}\rangle = e^{km}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dy\; e^{\boxed{ky - \frac{y^2}{2\sigma^2}}}$$

$$\frac{y^2}{2\sigma^2} - ky = \frac{1}{2\sigma^2}\left(y-k\sigma^2\right)^2 - \frac{1}{2\sigma^2}\left(k\sigma^2\right)^2$$

$$= e^{km}\frac{1}{\sqrt{\ \ }}\int_{-\infty}^{+\infty} dy\; e^{-\frac{1}{2\sigma^2}(y-k\sigma^2)^2}\; e^{+\frac{1}{2}k^2\sigma^2}$$

$$= e^{km}\, e^{k^2\sigma^2/2}$$

For any convex-up function $\varphi(x)$:



$\varphi(\langle X\rangle) \leq \langle\varphi(X)\rangle$ (Jensen inequality)

(1) Write $A_{ij} = \underbrace{\frac{1}{2}\left(A_{ij}+A_{ji}\right)}_{\substack{\text{symm}\\ A_s}} + \underbrace{\frac{1}{2}\left(A_{ij}-A_{ji}\right)}_{\substack{\text{antisym}\\ A_a}}$

${}^tA_s = A_s$

${}^tA_a = -{}^tA_a$

We need ${}^t\vec{x} \times A\vec{x}$ which is scalar and thus equal to its transpose ${}^t\vec{x}\,{}^tA\,\vec{x}$

↳ this vanishes for any antisym matrix. Indeed, take M antisym

$M_{ij} = -M_{ji}$; $\sum_{i,j} M_{ij}x_ix_j = -\sum_{i,j} M_{ji}x_ix_j = -\sum_{i,j} M_{ij}x_ix_j$

(2) $\vec{m} = \langle\vec{X}\rangle$

(3) $\frac{1}{\mathcal{N}} = \int dx_1\,dx_2\cdots dx_n\,\exp\left[-\frac{1}{2}\,{}^t\vec{x}A\vec{x}\right]$;

A hermitian (real symmetric) $\Rightarrow$ diagonalizable

$\Rightarrow \exists\ \Lambda$ diagonal and P

$\Lambda = P^{-1}AP$; $\Lambda = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

where $P = {}^tP^{-1}$

${}^tPP = P\,{}^tP = 1\!\!1 \rightarrow$ orthogonal matrix

Change variables: $\vec{y} = P^{-1}\vec{x}$

${}^t\vec{x}A\vec{x} = {}^t(P\vec{y})AP\vec{y} = {}^t\vec{y}\underbrace{\left({}^tPAP\right)}_{\Lambda}\vec{y} = \sum_{i=1}^{n}\lambda_i y_i^2$

↳ jacobian $|J| = \left|\frac{d\vec{y}}{d\vec{x}}\right| = |\det P| = 1$

$$\Rightarrow \frac{1}{\mathcal{N}} = \int dy_1 \cdots dy_n \, \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \lambda_i y_i^2\right] \quad ; \quad \text{all } \lambda_i \text{ should be } > 0$$

↳ A positive matrix

$$= \prod_{i=1}^{n} \sqrt{\frac{2\pi}{\lambda_i}}$$

$$= \frac{(2\pi)^{n/2}}{(\det A)^{1/2}}$$

(4) $$p(\vec{y}) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{ij} A_{ij}\, y_i y_j\right]$$ Take $\vec{m} = \vec{0}$

(5) $$\sum_j A_{ij}\langle x_j x_k\rangle = \sum_j \mathcal{N}\int dx_1 \cdots dx_n \, \underbrace{x_j x_k A_{ij}\, e^{-\frac{1}{2}\sum_{p,q} A_{pq} x_p x_q}}$$

$$= \mathcal{N}\int_{-\infty}^{+\infty} d\vec{x} \; x_k \left[-\frac{\partial}{\partial x_i} e^{-\frac{1}{2}\sum A_{pq} x_p x_q}\right]$$

$$= \underbrace{-\mathcal{N}\left[x_k e^{\cdots}\right]}_{0} + \mathcal{N}\int d\vec{x} \, \frac{\partial x_k}{\partial x_i} e^{-\frac{1}{2}\sum} \qquad \text{by parts}$$

$$= \delta_{ik}$$

$$\Rightarrow \langle x_j x_k\rangle = (A^{-1})_{jk}$$

(6) Same method $$\sum_j \underbrace{\langle x_i x_j\rangle}_{A^{-1}_{ij}} \left\langle \frac{\partial f}{\partial x_j}\right\rangle = \sum_j \int d\vec{x}\; A^{-1}_{ij} \frac{\partial f(\vec{x})}{\partial x_j} e^{-\frac{1}{2}A_{pq} x_p x_q} \mathcal{N}$$

$$= 0 - \sum_j A^{-1}_{ij} \mathcal{N}\int d\vec{x} \; f(\vec{x}) \Big(\sum_p A_{jp} x_p\Big) e^{-\frac{1}{2}\sum A_{pq} x_p x_q}$$ by parts

$$\sum_j A^{-1}_{ij} A_{jp} = \delta_{ip}$$

$$= + \sum_p \delta_{ip} \cancel{\mathcal{N}} \left\langle f(\vec{x})\, x_p \right\rangle$$

$$= \left\langle x_i f(\vec{x})\right\rangle$$

(7) 2ℓ factors in the product: $(2\ell-1)(2\ell-3)\cdots 1$

For $\ell=3$, 6 factors : $5\times 3 = 15$ terms

Scalar case : $\langle x^4\rangle = 3\langle x^2\rangle^2 = 3\sigma^4$

$\langle x^6\rangle = 15\sigma^6$

$\langle x^8\rangle = 7\times5\times3 = 105\sigma^8$

Proof : $\langle x_i x_j x_k x_p\rangle = \langle x_i f(\vec{x})\rangle$ for $f(\vec{x}) = x_j x_k x_p$

$= \sum_\ell \langle x_i x_\ell\rangle\left\langle\frac{\partial f}{\partial x_\ell}\right\rangle$; non zero for $\ell = j$ or k or p

$= \langle x_i x_j\rangle\langle x_k x_p\rangle + \langle x_i x_k\rangle\langle x_j x_p\rangle + \langle x_i x_p\rangle\langle x_j x_k\rangle$

(8) $p(x,y) = \mathcal{N} e^{\cdots}$: $A = \begin{pmatrix} 1 & 1 \\ 1 & 4\end{pmatrix}$; $A^{-1} = \frac{1}{4-1}\begin{pmatrix} 4 & -1 \\ -1 & 1\end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1\end{pmatrix}$

thus $\langle x^2\rangle = 4/3$

$\langle y^2\rangle = 1/3$

$\langle xy\rangle = -1/3$

$\langle x^2y^2\rangle = \langle x^2\rangle\langle y^2\rangle + \langle xy\rangle\langle xy\rangle \times 2 = \frac{4}{9} + 2\cdot\frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

$\langle x^4y^2\rangle = \langle xxxxyy\rangle = 3\langle x^2\rangle^2\langle y^2\rangle + 12\langle x^2\rangle\langle xy\rangle^2$

15 pairing

3 have $\langle y^2\rangle$ and $\langle x^4\rangle$

12 have $\langle xy\rangle\langle xy\rangle\langle x^2\rangle$

$= \cancel{3}\left(\frac{4}{3}\right)^2\frac{1}{\cancel{3}} + \overset{4}{\cancel{12}}\,\frac{4}{\cancel{3}}\,\frac{1}{9}$

$= \frac{16 + 16}{9} = \frac{32}{9}$

Extra $\langle x^2y^4\rangle = 3\langle x^2\rangle\langle y^2\rangle^2 + 12\langle xy\rangle^2\langle y^2\rangle = \cancel{3}\,\frac{4}{\cancel{3}}\,\frac{1}{9} + \overset{4}{\cancel{12}}\,\frac{1}{9}\,\frac{1}{\cancel{3}}$

15 pairings

3 have $\langle x^2\rangle, \langle y^2\rangle^2$

12 have $\langle xy\rangle^2\langle y^2\rangle$

$= \frac{4}{9} + \frac{4}{9}$

$= \frac{8}{9}$

(9) $$\left\langle e^{i\vec{k}\cdot\vec{x}}\right\rangle = \int dx_1 \cdots dx_n \; e^{i(k_1x_1+k_2x_2+\cdots k_nx_n)} \, \mathcal{N} \exp\left(-\frac{1}{2}\,{}^t\vec{x}A\vec{x}\right)$$

$$= \mathcal{N}\int d\vec{x}\; e^{i\,{}^t\vec{x}\vec{k} - \frac{1}{2}\,{}^t\vec{x}A\vec{x}}$$

the idea is to shift $\vec{x}$ by some vector $\vec{\delta}$ to be found:

$${}^t(\vec{x}-\vec{\delta})\,A\,(\vec{x}-\vec{\delta}) = {}^t\vec{x}A\vec{x} \underbrace{- {}^t\vec{\delta}A\vec{x} - {}^t\vec{x}A\vec{\delta}}_{\text{equal since scalar}} + {}^t\vec{\delta}A\vec{\delta}$$

$$= {}^t\vec{x}A\vec{x} - 2\,{}^t\vec{x}A\vec{\delta} + {}^t\vec{\delta}A\vec{\delta}$$

$$\Rightarrow -2\,{}^t\vec{x}A\vec{\delta} = -2i\,{}^t\vec{x}\vec{k} \quad ; \quad \text{take } i\vec{k} = A\vec{\delta} \quad ; \quad \vec{\delta} = iA^{-1}\vec{k}$$

$$\Rightarrow i\vec{k}\cdot\vec{x} - \frac{1}{2}\,{}^t\vec{x}A\vec{x} = -\frac{1}{2}\,{}^t(\vec{x}-\vec{\delta})A(\vec{x}-\vec{\delta}) \underbrace{+\frac{1}{2}\,{}^t\vec{\delta}A\vec{\delta}}_{+\frac{1}{2}(-1)\,{}^t\vec{k}\cancel{A^{-1}}\cancel{A}A^{-1}\vec{k}}$$

Change variables: $\vec{y} = \vec{x} - iA^{-1}\vec{k}$

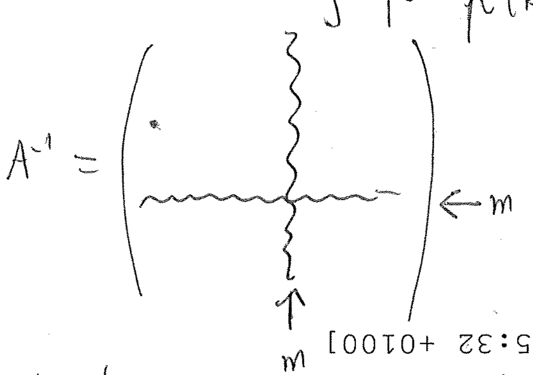
$$\left\langle e^{i\vec{k}\cdot\vec{x}}\right\rangle = e^{-\frac{1}{2}\,{}^t\vec{k}A^{-1}\vec{k}} \underbrace{\mathcal{N}\int d\vec{y}\; e^{-\frac{1}{2}\,{}^t\vec{y}A\vec{y}}}_{1}$$

(10) <u>Marginals</u> the charact funct of $p_{\setminus m} \equiv \int dx_1 \ldots dx_{m-1}\, dx_{m+1} \cdots dx_n \; e^{i(k_1x_1+\cdots k_nx_n)} \, p_{\setminus m}$ (without the $k_m x_m$ term)

$$\underbrace{\hat{p}_{\setminus m}(\vec{k}\setminus k_m)}_{k_1 \cdots k_{m-1}, k_{m+1} \cdots k_n} = \int e^{i(\vec{k}\cdot\vec{x}\setminus k_m x_m)} \, p(x_1 \ldots x_n)$$

$$= \hat{p}(\vec{k}) \quad \text{with } k_m = 0$$

the coupling matrix is read directly from $\hat{p}(\vec{k})$ and thus from A^{-1}



Erase line m and column m

↳ gives the coupling matrix of $p_{\setminus m}$

Trivial application to $A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$; $\frac{1}{3}\begin{pmatrix} \cancel{4} & \cancel{-1} \\ \cancel{-1} & 1 \end{pmatrix} \to \frac{1}{3}$

$\frac{1}{3}\begin{pmatrix} 4 & \cancel{-1} \\ \cancel{-1} & \cancel{1} \end{pmatrix} \to \frac{4}{3}$

(11) Linear combination of components

$\vec{z} = B\vec{x}$; $\langle e^{i\vec{k}\cdot\vec{z}} \rangle = ?$

$$\langle e^{i\vec{k}\cdot\vec{z}} \rangle = \langle e^{i\sum_\alpha k_\alpha z_\alpha} \rangle = \langle e^{i\sum_{\alpha,\beta} k_\alpha B_{\alpha\beta} x_\beta} \rangle = \langle e^{i\sum_\beta q_\beta x_\beta} \rangle$$

$$q_\beta = \sum_\alpha k_\alpha B_{\alpha\beta}$$

$$= e^{-\frac{1}{2}\sum_{i,j} q_i A^{-1}_{ij} q_j}$$

$$= e^{-\frac{1}{2}\sum_{i,j}\sum_{\alpha,\beta} A^{-1}_{ij} k_\alpha B_{\alpha i} k_\beta B_{\beta j}}$$

$$= e^{-\frac{1}{2}\sum_{\alpha,\beta} k_\alpha k_\beta \underbrace{\sum_{ij} A^{-1}_{ij} B_{\alpha i} B_{\beta j}}_{\tilde{A}_{\alpha\beta}}}$$

; $\boxed{\tilde{A} = B A^{-1} {}^tB}$

$$= e^{-\frac{1}{2}\underbrace{\sum_{\alpha\beta} k_\alpha \tilde{A}_{\alpha\beta} k_\beta}_{\text{quadratic}}}$$

quadratic $\Rightarrow$ the $\vec{z}$ is a gaussian vector
All its components are gaussian themselves

(12) Take $X \sim \mathcal{G}(0,1)$; X and X^2 are not independent
but $\langle X X^2\rangle - \langle X\rangle\langle X^2\rangle = 0$: 0 2-point correlation.
this provides a counterexample of 2 variables that are 2pt uncorrelated, but dependent

(13) Random walk :

a) $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_1+\eta_2 \\ \vdots \\ \eta_1+\cdots\eta_n \end{pmatrix} = B \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$ where B is the appropriate matrix $\Rightarrow \vec{x}$ is Gaussian

b) $p(x_1) \propto \exp\left(-\frac{x_1^2}{2}\right)$

$p(x_2|x_1) = p(\eta = x_2 - \eta_1) \propto \exp\left(-\frac{(x_2-x_1)^2}{2}\right)$

$\Rightarrow p(x_1,x_2) = p(x_2|x_1)\,p(x_1)$

$\propto \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2 - 2x_1x_2 + x_1^2\right)\right]$

$\propto \exp\left[-\frac{1}{2}\left(2x_1^2 + x_2^2 - 2x_1x_2\right)\right]$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$; then for 3 var: $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

Then $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ special

c) $\langle x_i x_j \rangle = \sum_{\alpha=1}^{i}\sum_{\beta=1}^{j} \langle \eta_\alpha \eta_\beta \rangle = \sum_{\alpha=1}^{\min(i,j)} \langle \eta_\alpha^2 \rangle = \min(i,j)$

Hence $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ yielding $\langle x_i^2 \rangle = i$ as it should

$AC = CA = \mathbb{1}$ (check)

d) Marginals from erasing lines & columns from $C = A^{-1}$.