Large deviations

1) What is the rate function ϕ for the large deviations of the sum of IID Gaussian variables with mean μ and variance σ^2? Propose three methods, including the direct exact calculation, to show that

$$\phi(s) = \frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2 . \qquad (1)$$

Figure 1 illustrates how the large deviation function $\phi(s)$ emerges from the pdf of the sample mean. How are the different parabolas shown on the right panel of Fig. 1 related?

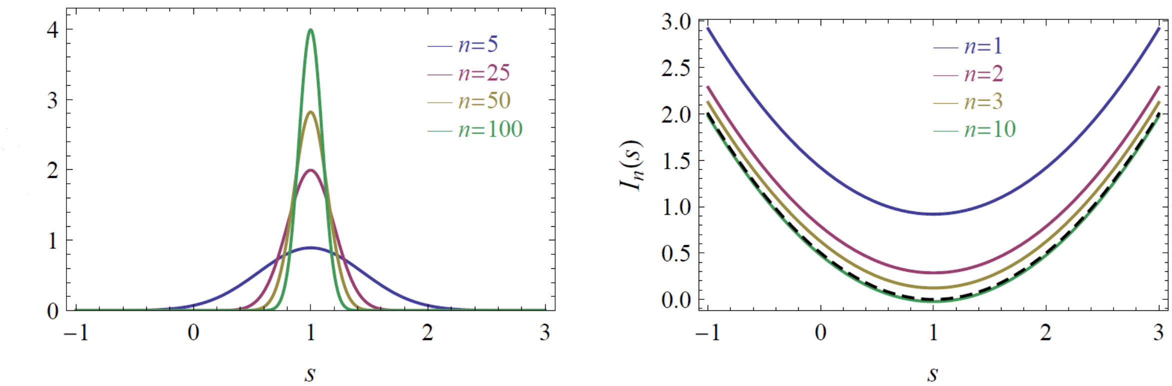

Figure 1: (Left) Probability density $p_n(s)$ of the empirical mean s of n independent Gaussian variables of mean 1 and variance 1. (Right) Plot of $I_n(s) = -n^{-1}\log p_n(s)$. The dashed line shows the parabola of equation $(s-1)^2/2$. From H. Touchette, *A basic introduction to large deviations*, `arXiv:1106.4146`

2) Same question for the sample mean of IID Bernoulli variables: $X = 0$ with probability $1-\alpha$, and $X = 1$ with probability α. While a direct calculation is possible, use the Sanov theorem to show that

$$\phi(s) = s\log\left(\frac{s}{\alpha}\right) + (1-s)\log\left(\frac{1-s}{1-\alpha}\right) . \qquad (2)$$

The emergence of the rate function when increasing the number of terms summed is illustrated in Fig. 2.

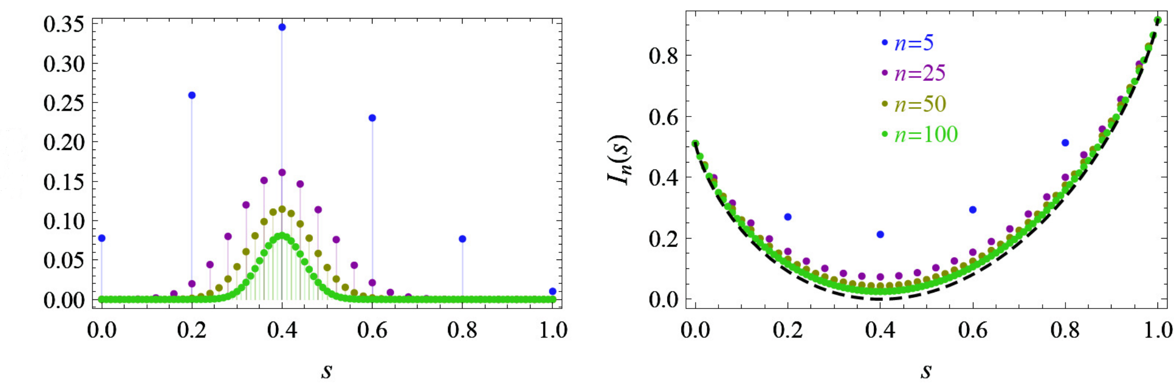

Figure 2: (Left) Same as Fig. 1 for the mean of n discrete Bernoulli variables with $\alpha = 0.4$. The dashed line shows the rate function $\phi(s)$. From H. Touchette, *A basic introduction to large deviations*, `arXiv:1106.4146`

3) Application. We toss 100 times a biased coin (probability head/tail is 10%/90%). What is (approximately, ie at the Sanov's level) the probability of getting 50 heads and 50 tails? Conversely, considering 100 tosses of a fair (unbiased) coin, what is the probability of getting 10% heads and 90% tails? Are these two probabilities equal? What does this illustrate?

4) Compute the rate function for the sum of IID variables distributed according to the double-sided exponential

$$p(\eta) = \frac{1}{2} e^{-|\eta|}, \quad \eta \in \mathbb{R}. \tag{3}$$

Compare the Sanov route with the application of Gärtner-Ellis theorem.

5) Consider a generic statistical physics problem at inverse temperature β, described by the canonical ensemble. Relate the cumulant generating function of the energy to the free energy. Show then that

$$\langle E^2 \rangle - \langle E \rangle^2 = kT^2 \, c_v \tag{4}$$

where E is the total energy and c_v the specific heat (at fixed volume).