Discrete random walks

A First results

1) Return probability of a random walk in 1D.
We consider a one-dimensional biased random walk on $\mathbb{Z}$, with a probability p to jump to the right, $1-p$ to jump to the left, at each time step. The walk starts from the origin O, and we are interested in the return probability to O, denoted R. We also introduce $R_{\rightarrow}$, the probability to visit site O after an arbitrary amount of time, starting from the nearest site on its right (site 1). Likewise, we denote $R_{\leftarrow}$ the probability to visit O, starting from the site on its left (site -1).

a) Relate R to $R_{\rightarrow}$ and $R_{\leftarrow}$.

b) By partitioning on the number of visits to site 1, compute $R_{\rightarrow}$. Similarly, obtain $R_{\leftarrow}$.

c) What is the expression of R? What does this expression become for a symmetric walk ($p = q = 1/2$)?

2) Exit time from an interval.
The random walker described above is introduced at $t = 0$ at a site ℓ inside the interval $[0, N]$, *ie* $0 < \ell < N$. The walker is removed as soon as it reaches one of the extremities 0 or N. We denote $T(\ell)$ the random variable associated to the lifetime of the walker.

a) By partitioning on the first step of the random walk, write a finite difference equation obeyed by the probability $P(T(\ell) = n + 1)$.

b) What is then the equation fulfilled by the mean life time $\langle T(\ell)\rangle$? What are the associated boundary conditions?

c) Solve the above equation. Distinguish the cases $p \neq q$ and $p = q = 1/2$. What is found in the case of a symmetric walk?

B The generating function formalism

1) The generating functions $P(\mathbf{s}\,|\,\mathbf{s_0};\xi)$ and $F(\mathbf{s}\,|\,\mathbf{s_0};\xi)$
For a discrete time random walk, we define the following probabilities

- $P_n(\mathbf{s}\,|\,\mathbf{s_0})$: probability to be at site $\mathbf{s}$ after n jumps, given that the walk started at site $\mathbf{s_0}$;
- $F_n(\mathbf{s}\,|\,\mathbf{s_0})$: probability to be at site $\mathbf{s}$ *for the first time* at step n, given the walk started at site $\mathbf{s_0}$.

We also define the associated generating functions by

$$P(\mathbf{s}\,|\,\mathbf{s_0};\xi) \equiv \sum_{n=0}^{+\infty} P_n(\mathbf{s}\,|\,\mathbf{s_0})\,\xi^n, \quad \text{and} \quad F(\mathbf{s}\,|\,\mathbf{s_0};\xi) \equiv \sum_{n=0}^{+\infty} F_n(\mathbf{s}\,|\,\mathbf{s_0})\,\xi^n.$$

In the remainder, we adopt the convention : $P_0(\mathbf{s}\,|\,\mathbf{s_0}) = \delta_{\mathbf{s},\mathbf{s_0}}$ and $F_0(\mathbf{s}\,|\,\mathbf{s_0}) = 0$.

a) Write the normalization conditions fulfilled by the probabilities $P_n(\mathbf{s}\,|\,\mathbf{s_0})$ and $F_n(\mathbf{s}\,|\,\mathbf{s_0})$. Introduce $R(\mathbf{s}\,|\,\mathbf{s_0})$, the probability to visit $\mathbf{s}$, starting from $\mathbf{s_0}$, after an arbitrary number of steps.

b) By partitioning on the instant of the first visit to site $\mathbf{s}$, write a relation between the $P_n(\mathbf{s}\,|\,\mathbf{s_0})$ and the $F_n(\mathbf{s}\,|\,\mathbf{s_0})$.

c) Write then the generating function $F(\mathbf{s}\,|\,\mathbf{s_0};\xi)$ as a function of $P(\mathbf{s}\,|\,\mathbf{s_0};\xi)$.

2) Characterization of recurrent random walks
State a necessary and sufficient condition on the generating function $P(\mathbf{s}\,|\,\mathbf{s_0};\xi)$ such that the return probability to the starting site $\boldsymbol{s}_0$ be 1.

3) Mean number of returns to a given site
For a walk starting from $\mathbf{s_0}$, we introduce the boolean variable $I_n(\mathbf{s}|\mathbf{s_0})$, equal to 1 if site $\mathbf{s}$ is occupied after n steps, and equal to 0 otherwise. Show that $P(\mathbf{s}\,|\,\mathbf{s_0};1)$ represents the mean number of visits to site $\mathbf{s}$ for a walk starting at site $\mathbf{s_0}$. In particular, what can we say concerning the mean number of returns to the starting point $\mathbf{s_0}$ for a recurrent walk? A non recurrent walk?

4) Conditional mean first passage times
For a walk starting at site $\mathbf{s_0}$, we define $\tau(\mathbf{s}\,|\,\mathbf{s_0})$ as the mean number of steps until the (first) visit to site $\mathbf{s}$, given that this site is indeed reached during the walk. Relate $\tau(\mathbf{s}\,|\,\mathbf{s_0})$ to the generating function $F(\mathbf{s}\,|\,\mathbf{s_0};\xi)$.

5) Application to the one-dimensional biased random walk
Our interest goes to the 1D biased random walk, with probabilities p and $q = 1-p$ for right and left jumps respectively. What are the generating functions $P(\mathbf{s_0}\,|\,\mathbf{s_0};\xi)$ and $F(\mathbf{s_0}\,|\,\mathbf{s_0};\xi)$? Deduce from this

- the probability to return to site $\mathbf{s_0}$, $R(\mathbf{s_0}\,|\,\mathbf{s_0})$;
- the mean number of returns to the starting site $\mathbf{s_0}$;
- the conditional mean first return time $\tau(\mathbf{s_0}\,|\,\mathbf{s_0})$.

It is convenient to use here

$$\frac{1}{\sqrt{1-4y}} = \sum_{n=0}^{\infty} \binom{2n}{n} y^n. \tag{1}$$

C Translationally invariant random walks

In this section, we apply the above formalism to the study of discrete random walks, that are invariant by translation on a cubic lattice in dimension d. We denote $P_n(\boldsymbol{\ell})$ the probability to be at position $\boldsymbol{\ell}$ after n steps, given that the walker started from $\mathbf{0}$; $p(\boldsymbol{\ell}-\boldsymbol{\ell}')$ is for the probability to jump from position $\boldsymbol{\ell}'$ to $\boldsymbol{\ell}$ in one step. Besides, we adopt the convention $P_0(\boldsymbol{\ell}) = \delta_{\boldsymbol{\ell},\mathbf{0}}$.

1) Discrete Fourier transform and structure factor

a) Write the recurrence relation between P_{n+1} and P_n.

b) Introducing the discrete Fourier transforms

$$\widetilde{P}_n(\boldsymbol{k}) \equiv \sum_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{k}}\, P_n(\boldsymbol{\ell}) \qquad \text{and} \qquad \lambda(\boldsymbol{k}) \equiv \sum_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{k}}\, p(\boldsymbol{\ell}), \tag{2}$$

(the latter being referred to as the "structure factor" of the random walk), compute $\widetilde{P}_n(\boldsymbol{k})$. By inverse Fourier transformation, find $P_n(\boldsymbol{\ell})$.

c) From the previous question, obtain an integral representation of the generating function

$$P(\boldsymbol{\ell};\xi) \equiv \sum_{n=0}^{\infty} P_n(\boldsymbol{\ell})\xi^n. \tag{3}$$

2) 1D biased random walk
We go back to the 1D biased walk with probabilities p and $q = 1-p$. Calculate the generating function $P(\boldsymbol{\ell};\xi)$, and then the generating function of first passage probabilities:

$$F(\boldsymbol{\ell};\xi) \equiv \sum_{n=0}^{\infty} F_n(\boldsymbol{\ell})\xi^n. \tag{4}$$

What is then the probability to visit, being patient enough, an arbitrary site on the lattice?

3) Pólya's theorem
We consider a d-dimensional isotropic random walk on $\mathbb{Z}^d$ (with jumps to the nearest neighbors, so-called "Pólya random walk").

a) Making use of the following representation of modified Bessel functions with integer index n

$$I_n(z) = I_{-n}(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\mathrm{d}\theta, \tag{5}$$

show that the generating function $P(\boldsymbol{\ell};\xi)$ can be cast as

$$P(\boldsymbol{\ell};\xi) = \int_0^{\infty} e^{-t}\prod_{j=1}^{d} I_{|\ell_j|}(t\xi/d)\,\mathrm{d}t, \qquad \text{where } \ell_j \text{ is for the } j^{\text{th}} \text{ Cartesian component of the vector } \boldsymbol{\ell}. \tag{6}$$

b) Knowing the asymptotic expansion

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}\left\{1 - \frac{4n^2-1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right\} \quad \text{for } z \to \infty, \tag{7}$$

show that if $d = 1$ or $d = 2$, the walk is recurrent, while for $d \geq 3$ it is not ("Pólya's theorem").