The Bohigas-Giannoni-Schmit Conjecture

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1 Introduction

In their seminal 1984 paper [11] (see also [12]), Bohigas, Giannoni and Schmit stated a conjecture describing the statistical properties of chaotic spectra that was formulated as follows:

Spectra of time reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE (alternative stronger conjectures that cannot be excluded would apply to less chaotic systems, provided that they are ergodic).

Thirty years later this conjecture, although still not proven, is supported by a vast host of numerical studies, as well as what, in the judiciary world, would be referred to as “a convergent set of convincing evidences”. Its very general validity, in spite of some rather well understood limitations, is now largely accepted; and together with its various natural extensions (including time reversal non invariant systems, systems with spins, wave function properties, etc..) it forms the basis of a statistical approach to “simple” quantum systems, somewhat in the same spirit as the maximum entropy principle would for statistical physics at large. Stating a universal property of chaotic systems – beyond the fact that its classical analog is chaotic, nothing is said about the system – the BGS-conjecture has found applications in many fields of physics including atomic physics, nuclear physics, mesoscopic physics, etc., and has thus taken a significant place in the toolkit of theoretical quantum physicists.

The goal of this short review is therefore to provide some introduction to the meaning and implications of the BGS-conjecture. The body of the text will be divided in three parts. Section 2 will mainly be devoted to a more detailed discussion of the content of the conjecture, as well as of the context within which it has emerged. In section 3, the various attempts to prove, or justify, the BGS-conjecture will be briefly surveyed. Finally, an overview of the various applications of the conjecture will be found in section 4, as well as some concluding remarks.

2 Content of the BGS-conjecture

The kind of systems the BGS-conjecture aims to describe are simple quantum mechanical systems for which one can define a classical limit. The word “simple” here has to be understood as opposed to “complex”, itself associated with the existence of many degrees of freedom and with the interaction between phenomena taking place at different scales.

Typically one has therefore in mind low dimensional systems described by smooth Hamiltonians or billiards, and we shall use the latter for illustration. A two-dimensional quantum billiard is associated with a Hamiltonian reduced to its kinetic term

\[ H_K = -\frac{\hbar^2}{2m} \Delta, \]

within some region \( \mathcal{D} \) of the plan \( \mathbf{r} = (x, y) \), and some boundary conditions on the boundary \( \partial \mathcal{D} \) of the billiard. In the classical limit, this system is described by free propagation within the billiard \( \mathcal{D} \), and specular reflection on its boundary. Two examples of billiards are displayed in Fig. 1.

2.1 Chaos

Although such a two dimensional billiard is definitely a “simple system”, the classical motion within the billiard can be rather complicated, and can even be chaotic. One can compare in
this respect the circular billiard of Fig. 1a, which is actually an integrable system, with the Sinai billiard of Fig. 1b, for which the motion displays the strongest form of chaos.

Intuitively speaking, such a concept of chaos is associated with the notion of sensitivity to initial conditions and exponential divergence of neighbor trajectories. More formally, one should however rather refer to the hierarchy of chaos, which, starting from ergodicity – which by itself is not yet fully associated with chaos – leads to mixing, K and B-system, and finally axiom A, or Anosov, systems. Appendix A gives a brief introduction to these various notions. The conjecture itself, although it contains the mention of K-system, does not insist that the system should be sufficiently high in this hierarchy to apply (as is illustrated by the “alternative stronger conjectures” mentioning only ergodicity). We shall see in the next section that the relevant notion is more the absence of time scales in the exploration of phase space (or the fact that these time scale a short enough with respect to Heisenberg time).

### 2.2 Random Matrices

What the BGS-conjecture does is that it reveals the very strong link that exists between chaos, which is a concept of classical mechanics, and random matrix theory, which is a model for the statistical description of spectra and wave functions of quantum systems.

Random matrices were introduced in the fifties by Eugene Wigner in the context of nuclear physics. His goal at that time was to interpret the statistical properties of slow-neutron resonances, which could be understood as highly excited states of nuclei. Because nuclei typically contain a large number of protons and neutrons interacting through complicated forces, Wigner postulated that, from a statistical point of view, the Hamiltonian describing the nuclei for an energy range significantly above the ground state should not differ significantly from a ensemble of random matrices provided this ensemble respect the following constraints:

- The quantum evolution should be unitary, implying that the corresponding Hamiltonian has to be hermitian (and if real, symmetric).

- All symmetries should be taken into account, and especially the symmetry under time reversal.
• Beyond these symmetries, no information is contained in the ensemble, and in particular no direction of the Hilbert space plays a specific role.

Once these constraints were fulfilled Wigner postulated that the details of the matrices distribution law would not matter much, and that one could as well consider the simple case for which matrix elements are independent random quantities. It turns out that this constraints completely specify the random matrix ensembles that one needs to construct, and leads to the three Gaussian Ensemble of Wigner – GOE (Orthogonal), GUE (Unitary) and GSE (Symplectic) – that we describe now.

A matrix ensemble is specified by both the set $\mathcal{H}$ of matrices that are considered, and by the probability density defined on this set. The set of matrices depends on symmetry with respect to time reversal:

- If the system under consideration is not invariant under time reversal symmetry

  $$\mathcal{H} = \{ N \times N \text{ hermitian matrices} \} .$$

  (This corresponds to the Gaussian Unitary Ensemble (GUE)).

- If the system is invariant under time reversal symmetry, and either has an integer total quantum momentum or either rotational or inversion symmetry

  $$\mathcal{H} = \{ N \times N \text{ real symmetric matrices} \} .$$

  (This corresponds to the Gaussian Orthogonal Ensemble (GUE)).

- If the system is invariant under time reversal symmetry, and has a half-integer total quantum momentum without rotational or inversion symmetry

  $$\mathcal{H} = \{ 2N \times 2N \text{ quaternion real matrices} \} .$$

  (This corresponds to the Gaussian Symplectic Ensemble (GSE)).

Noting $\beta$ the number of real parameters necessary to define one matrix element ($\beta = 1$ for GOE (real matrix elements), $\beta = 2$ for GUE (complex matrix elements), and $\beta = 4$ for GSE (quaternion real matrix elements)), the probability density takes a rather similar form since for any $H \in \mathcal{H}$,

$$P_{N\beta}(H)dH = \mathcal{N}_{N\beta} \exp \left[ -\frac{\text{Tr}(H^2)}{2v^2} \right] dH ,$$

with $\mathcal{N}_{N\beta}$ a normalization constant that depends both on $N$ and $\beta$ and $v$ a parameter that fixes the energy scale. It has to be born in mind however that in spite of the similarity of the expressions of $P_{N\beta}(H)$ for different $\beta$ they are actually rather different objects since they apply to different kind of matrices. In particular the measure $dH$ means $\prod_{i \leq j} dH_{ij}$ in the orthogonal case but $\prod_i dH_{ii} \prod_{i < j} dH_{ij} dH_{ij}^I$ (with $H_{ij}^R$ and $H_{ij}^I$ the real and imaginary parts of $H_{ij}$), in the unitary case, and again something different in the symplectic one. The names of the matrix ensembles originate from the fact that the density probability $P_{N\beta}(H)$ is invariant under any orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) transformation, which expresses the fact that no direction of the Hilbert space plays a particular role.

To each of these random matrix ensembles (which are usually considered in the $N \to \infty$ limit) correspond definite predictions for various spectral statistics such as $k$-point correlation function, the nearest neighbor spacing, or the $\Sigma^2$ statistics. In appendix B we review briefly the most popular of these statistics and give some of the expressions that can be derived for them within the Gaussian Ensemble modeling. Somewhat improperly, since other kind of
matrix ensembles could be, and actually are, designed, we shall refer from now on to Gaussian Ensembles as “the Random Matrix Ensemble”.

Here we just insist on the two important features that characterize random matrix ensembles, namely level repulsion and level rigidity. Level repulsion is a short distance effect, which takes place on a scale shorter than the mean level spacing, and which states that the probability of finding levels very close one from each others is small. The tendency of levels to repel each other at short distances increases with $\beta$, and is therefore larger for GUE than for GOE, and again larger for GSE than for GUE. Level rigidity on the other hand is a large energy scale property (ie energy scale much larger that the mean level spacing). It states that in an energy range which contains on average a number $M$ of level the fluctuation of the number of level in this range remains small even when $M$ becomes large (in practice grows logarithmically with $M$, and for instance not as $\sqrt{M}$ as could be expected for a random process).

2.3 Complexity vs Chaos

As mentioned above, what Wigner had in mind when he introduced his Random matrix ensembles was that the relevant notion to motivate their use was complexity. The successes of the random matrix modeling in nuclear physics however made it clear that even beyond this field, spectral statistics were an appropriate tool to characterize different kinds of quantum systems.

In his 1973 paper on “Regular and irregular spectra” [47], Percival built on on this perspective, and introduced the notion that spectral statistics could be one way to distinguish the quantum mechanics of classically chaotic system from the one of classically regular (integrable) ones. At a very general level he suggested that since classical dynamics was a limit of quantum dynamics, the integrable or chaotic nature of the classical dynamics should have consequences on the corresponding spectra. More precisely, he argued that when integrable systems should show no level repulsion (their eigenstates live on invariant tori which do not overlap in phase space, and thus in some sense ignore each other), chaotic states should be in contact with one other (since the chaotic motion mixes everything) and thus should show some degree of level repulsion, in analogy with RMT.

In 1977, Berry and Tabor [8] further showed (for more than one degree of freedom systems and with the exception of pathological cases such as the harmonic oscillator) that not only there is no level repulsion in integrable systems but in the semiclassical limit they display the same statistical properties as a pure random process, ie as if one was just drawing the levels independently from each other. Berry and Tabor derivation of this results has now recently become a full fledged theorem [40].

The coup de g`enie of Bohigas Giannoni and Schmit was to realize that the difference between the spectral statistics of integrable and chaotic systems was not just that the former could be modeled by a set of independent numbers when the latter shows “some” correlation (and in particular some degree of level repulsion), but that the spectral statistics of chaotic systems could be fully, and in every details, described by the random matrix ensembles of Wigner. As illustrated in Fig. 2 they based their conclusions on a detailed numerical study of a few chaotic billiards (the Sinai and Bunimovitch billiard, which are both Anosov system), numerical studies that were confirmed by a vast number of other works.

By doing so, they showed that contrarily to what Wigner had in mind, chaos rather than complexity was the relevant concept to associate with random matrices. This significantly enlarged the universality class to which this kind of modeling could be applied, and, as we shall illustrate in section 4, to the number of field in physics within which one could apply a random matrix description.
3 Justifications of the BGS-conjecture

Up to this day, most of the confidence one has in the validity of the random matrix description stems from the fact that – in addition to the Sinai and Bunimovitch billiards studied by Bohigas Giannoni and Schmit – it has been verified numerically on a vast variety of simple systems, including the Hydrogen Atom in magnetic field [58, 20], the hyperbola billiard [49], and various other kind of systems.

A few attempt have been made however to “prove” the BGS conjecture. Although none of these proofs can pretend to be fully convincing – even within the physicist acception of the notion of proof – they shade some light on the mechanisms underlying the BGS-conjecture. We try to summarize (very briefly) here their content.

3.1 Non-linear $\sigma$-model

Non-linear $\sigma$-models were initially introduced in solid state physics in the context of disordered system, with the aim to go beyond the weak disorder / diffusive approximation and to describe localization effects. In this respect it has proved to be a very useful and effective tool.

In 1996, Andreev and coworkers made an attempt to apply this approach to chaotic, rather than diffusive, systems, having in mind very specifically to derive in this way the BGS conjecture.

Non-linear $\sigma$-models are rather technical objects, and it is unrealistic to enter in any significant way into their formalism. We shall thus here limit ourselves to a kind of literary description of [3, 4], without actually writing down any equation.

The starting point of this approach, as for diffusive system, is to write down an exact form of the two-point correlation generating function in terms of a functional integral involving both bosonic and fermionic variable. Performing an energy average (instead of the disorder average done for disordered systems) produces a quartic (interaction-like) term in the resulting action, which can be decoupled through a Hubbard-Stratonovitch transformation.

This description is up to this point exact. One then performs first a steepest descent approximation in the large $N$ limit (with $N$ the number of states contained within the Thouless energy), and a semiclassical approximation where the commutators are replaced by Poisson brackets. In this way, the two-point correlation function can be expressed in terms of the...
Perron-Frobenius operator of the classical motion, and if there is a gap between the lowest eigenvalue of this operator (which corresponds to the ergodic distribution) and the next lowest one, one recovers, for time reversal non invariant systems, the corresponding GUE result in the form that can be obtained from a supersymmetric calculation of this quantity.

Beyond its very technical character, this derivation suffers from a couple of drawbacks. The first one, which could have been only technical, is that the spectra of the Perron-Frobenius operator (and thus the gap between the two lowest states) depends significantly on the space of function to which it is applied, and in the midst of the various approximation performed during the derivation it is not clear that the regularization done makes it possible to determine unambiguously this space.

More troubling is that although this approach leads to the correct result for GUE, it seems to fail capturing the interference effects associated with time reversal invariance in the GOE case \[39\], leading (under the same set of approximations as for GUE) to an incorrect prediction for the two-point correlation function.

3.2 Semiclassical trace formula

Another approach that has been used to address the BGS conjecture is through the use of the Gutzwiller semiclassical trace formula. Since the BGS conjecture is making a link between a classical property of a system (the fact that its classical dynamics is chaotic) and some of its quantal ones (the spectral statistics), it is indeed quite natural to imagine that this link is founded on a semiclassical approximation. For chaotic systems, a natural candidate is the Gutzwiller trace formula which relates the density of state \(d(\epsilon) = \sum_n \delta(\epsilon - \epsilon_n)\) (with \(\epsilon_0, \epsilon_1, \cdots\) the quantum eigenlevels) of the quantum system to a sum over all periodic orbit \(j\) of its classical analog. More specifically \(d(\epsilon)\) can be split into a smooth part \(\bar{d}(\epsilon)\) and an oscillating one \(d^{osc}(\epsilon)\) and the Gutzwiller trace formula reads

\[
d^{osc}(\epsilon) \equiv \frac{1}{\pi \hbar} \sum_j \frac{1}{\sqrt{\det(M_j - 1)}} \cos \left( \frac{1}{\hbar} S_j - \nu_j \frac{\pi}{2} \right),
\]

(3)

where \(S_j, M_j\) and \(\nu_j\) are classical quantities associated with the periodic orbit \(j\) at energy \(E\). More precisely \(S_j = \oint_{\text{orbit } j} p \, dr\) is the action integral along the orbit \(j\), \(M_j\) is the monodromy matrix describing the stability of the linear motion near the orbit (the more unstable the orbit, the larger \(\det(M_j - 1)\)), and the Maslov index \(\nu_j\) is an integer related to local winding around the orbit.

The first work making use of the Gutzwiller trace formula to compute a spectral statistics was performed by Berry in his 1985 paper \[7\]. His main focus in this paper was the Dyson Metha function \(\Delta(L)\) Eq. (23), but he provides also a discussion of the form factor \(K(\tau)\) Eq. (14) which is somewhat simpler and that we summarize with some details now.

Inserting the semiclassical expression Eq. (3) in the definition Eq. (13)-Eq. (14) of the form factor gives

\[
K(\tau) \simeq \left( \sum_i \sum_j^+ A_i A_j \cos\{(S_i - S_j)/\hbar\} \delta \left( \tau - \frac{1}{2}(\tau_i + \tau_j) \right) \right),
\]

(4)

in which the + on the summation denotes restriction to positive traversals, \(\tau_{i,j} \equiv T_{i,j}/\hbar\) and \(A_{i,j} \equiv \left[ \pi \sqrt{\det[M_{i,j} - 1]} \right]^{-1}\).

In the semiclassical regime \(\hbar \to 0\), the factor \(\cos\{(S_i - S_j)/\hbar\}\) is usually very rapidly oscillating, and thus averaging to zero, unless the variation of \((S_i - S_j)\) with energy is small.
(i.e. \( \sim \hbar \)), that is
\[
\frac{d}{dE} [S_i - S_j] \equiv (T_i - T_j) \simeq \hbar .
\] (5)

This implies in particular that \( K(\tau) \) is dominated by periodic orbits of period \( T \simeq \hbar \tau \).

One can then distinguish three ranges of \( \tau \). For large \( \tau \) (i.e. \( \tau > 1 \), corresponding to orbit lengths of order \( \hbar/\Delta \)), long orbits are involved and their exponential proliferation makes it relatively easy to fulfill Eq. (5) for a large number of orbit pairs. A bootstrapping argument has been used by Berry to show that \( \lim_{\tau \gg 1} K(\tau) = 1 \).

For \( \tau \) small enough on the other hand, only reasonably short periodic orbits are involved, and there are such a small number of them that the only way to fulfill Eq. (5) is to choose \( i = j \). In this “diagonal approximation”, the form factor reduces to a simple (rather than double) sum
\[
K(\tau) \simeq \left\langle \sum_i |A_i|^2 \delta (\tau - T_i/\hbar) \right\rangle ,
\] (6)
in which no oscillating term exists any more.

For very small \( \tau \), that is \( \tau \simeq \tau_{\min} \overset{\text{def}}{=} T_{\min}/\hbar \), with \( T_{\min} \) the shortest period of the system, only a very small number of orbit can enter into the game and \( K(\tau) \) will depend on their specific properties (period en stability), and for \( \tau < \tau_{\min} \) one actually has \( K(\tau) \equiv 0 \). This is the non universal regime which of course cannot correspond to the random matrices prediction.

For \( \tau \) sufficiently above \( \tau_{\min} \) that the periodic orbits of period \( \hbar \tau \) cover uniformly the phase space, it has been shown by Hannay and Ozorio de Almeida [24] that
\[
\Phi_D(\tau) = \sum_i |A_i|^2 \delta (\tau - T_i/\hbar) \to \frac{T}{4\pi} .
\] (7)

Thus, if \( \tau \) is sufficiently above \( \tau_{\min} \), but yet sufficiently small that the diagonal approximation Eq. (6) is accurate, one has
\[
K_{TNRI}(\tau) \simeq \tau \quad \tau_{\min} \ll \tau \ll 1 .
\]
which is exactly what GUE predicts for the form factor in this range (cf. Eq(17)).

The same analysis can be performed for time reversal symmetric systems. The only difference is that in that case each periodic orbit \( i \) is associated with a time reversal partner \( i^* \) which action \( S_{i^*} \) is the same as \( S_i \). The diagonal approximation then reads
\[
K(\tau) \simeq 2\Phi_D(T),
\]
since one should pair \( i \) both with itself ans with its time symmetric \( i^* \). This leads to
\[
K_{TRI}(\tau) \simeq 2\tau \quad \tau_{\min} \ll \tau \ll 1 .
\]
which again is exactly what GOE predicts for the form factor in this range.

Thus, in the range \( \tau_{\min} \ll \tau \ll 1 \), which is such that both periodic orbits covers the phase space uniformly and the diagonal approximation is valid, periodic orbit theory and random matrix modeling give the same answer. The universal character of RMT is born out by the uniform covering of phase space by periodic orbits (which indeed leaves little space for system specific informations). Going back to energy representation (i.e. for instance to the two-point correlation function \( R_2(s) \)), this range of validity corresponds to \( \Delta \ll \delta \epsilon \ll E_{Th} \) where \( \delta \epsilon = s\Delta \) and \( E_{Th} \equiv \hbar/T_{\min} \) is the Thouless energy.

\(^1\)Doing this, there is a double counting of self-retracing orbits since they are their own time reversal symmetric partner. However the number of self-retracing orbits is small enough to make this effect marginal.
If one wants to address smaller energy range, longer trajectories come into play, and the diagonal approximation cannot be used any more. For chaotic systems, however, the pairs of orbits that actually contribute to the spectral statistics have a remarkable structure that allows to classify them. Going on with the example of the form factor, Sieber and Richter [51, 50] have shown that the pair of orbits contributing to $K(\tau)$ need to include the kind of “loop” sketched in Fig. 3 (the role of these loops in the connection between the semiclassical and the RMT descriptions of quantum transport through a ballistic quantum dots was also analyzed in [48]). Such a loop involves a near crossing from which a small variation makes it possible to switch from the clockwise to the anticlockwise direction of travel of the loop. For time reversal symmetric systems, pairs of periodic orbits which except for such loop follow each other closely can be shown to have almost exactly the same actions and period. Such pairs of periodic orbits therefore contribute to the form factor for time reversal symmetric systems (but not for time reversal non-symmetric ones). Taking into account the pairs of orbits with one such loop, together with an assumption of uniform covering of phase space by the orbits, Sieber and Richter [51] were able to recover the first non-linear correction to $K(\tau)$ predicted by GOE, as well as to justify semiclassically the absence of such a non-linear correction for GUE (cf Eq. (17)). Including an arbitrary number of loops makes it furthermore possible to reconstruct the small $\tau$ expansion of $K(\tau)$ of GOE to arbitrary order [53]. Further works, involving field theoretical resummation techniques and bootstrap consideration made it furthermore possible to recover in this way semiclassically all the spectral statistics predicted by random matrices [43, 44, 42].

Figure 3: Sketch of a loop (in configuration space) for a pair of trajectories which, for time reversal symmetric systems, have very similar period. Pairs of periodic orbits containing such loops give corrections to the diagonal approximation Eq. (6) which reconstruct the GOE results.

This semiclassical analysis based on the Gutzwiller trace formula and the Richter-Sieber loop expansion shows that there is indeed a strong link between classical chaos and random matrix theory, and for instance makes it possible to compute corrections to RMT predictions associated with a finite Ehrenfest time [18]. It cannot however be considered as a proper proof of the BGS conjecture for two reasons. The first one is that although a set of terms has been identified which when taken into account reproduces the random matrix theory statistics, there is no proof yet that all other contribution can safely be neglected (this is most presumably true since one indeed recover the expected RMT results, but the induction here rather goes the opposite way). A more serious limitation however is that being semiclassical in essence, the “range of validity” of this approach cannot be extended beyond the one of the semiclassical approximation. It turns out that there are many indications that in general, the Gutzwiller trace formula (or its improved versions) does not converge for orbit lengths that can resolve the mean level spacing. Thus level statistics such as the nearest neighbor distribution, or the short range behavior of the two-point correlation function, are in principle out of the scope of this approach; or in the most optimistic version a complete proof should include as a lemma a proof that the Gutzwiller trace formula converges for such long orbits.

It remains that this semiclassical “loop theory” initiated by Sieber and Richter provides, for physicists, a sound semiclassical link between classical chaos and random matrix theory. In
particular it demonstrates that one should not expect systematic “semiclassical corrections” to RMT based on the intrinsic structure of chaotic classical dynamics. Conversely, as the necessary ingredient underlying this link is identified (the uniform covering of phase space with trajectories containing the relevant loops), conditions under which deviation to RMT can be expected can be spelled out [18, 17]. More generally this approach offers an insightful point of view on the BGS conjecture.

3.3 Further discussion: the ergodicity argument

As we have seen, the approaches based on the non-linear $\sigma$-model and on the Gutzwiller trace formula provide (especially the second one) significant insights into why the BGS conjecture apply, but cannot be taken as a definitive proof of this conjecture. It may therefore be useful to provide another, even less rigorous, argument in favor of the BGS conjecture.

The basic property at work for this hand-waving argument is the ergodicity of the random matrix ensembles [46], namely the fact that if one selects at random one matrix in, say, the GOE ensemble, then, in the large $N$ limit, one knows that, with probability one, spectral averages taken on this particular matrix are equivalent to ensemble averages over the full GOE ensemble.

It is presumably not easy to implement this ergodicity argument into an honest “proof” of the BGS conjecture as, when one considers a particular chaotic system (eg. a specific billiard [49], or the Hydrogen atom in a magnetic field [58]) the latter is usually not taken at random, and in any case not with the probability Eq. (2). This however somewhat changes the perspective as it shows that in some sense RMT behavior is the “normal” behavior, and what should be explained are rather the exceptions to RMT. Or in other words it makes it possible to rephrase the question of whether a RMT modeling should apply as whether the particular system under consideration belongs to a zero-measure subset of the ensemble that might have non-generic spectral statistics.

In that sense, chaotic systems can be seen as systems on which one has no a priori information, and therefore no reason to consider them as non-generic. Integrable systems on the other hand are such that their classical limit is extremely constrained because of the existence of global conserved quantity, and it is not surprising that in the semiclassical regime, these classical constraints are going to make the quantum system itself non-generic, and thus non-RMT. In the same spirit, arithmetic billiards [10] display the strongest form of classical chaos but their quantum mechanics is characterized by an infinite set of conserved quantity (the Hecke operators which are commuting with the Hamiltonian). They can be expected to be non-generic, and are indeed shown to display non-RMT statistics.

In this perspective, the case of mixed system is presumably quite illuminating. Mixed systems are characterized by the coexistence of chaos and regularity, that is of regions of phase space where local conserved quantities exist and others where none can be defined. As suggested by Percival [47], this difference of nature of the dynamics in different part of the phase space should be associated with a separation of the quantum states in two classes, one associated with the regular motion (the integrable states) and one with the chaotic motion (the chaotic states). This separation in two classes can actually be performed [15]. However, even if the chaotic states are associated with a part of the phase space which is chaotic, they do not necessarily display the Gaussian Ensemble statistics of RMT [14].

The reason for this is that even if the motion in the chaotic part of the phase space is ergodic (essentially by definition), the way this exploration takes place is usually not structureless. The fact that some local constant of motion are only slightly broken generically implies the existence of partial barriers, which do not completely block the transport (and thus do not prevent ergodicity, which is a long time property) but may slow it down considerably.
Thus in mixed systems, what is typically seen is that the chaotic part of the phase space is not explored uniformly, but rather trajectories are trapped for some time in some region of phase space, then jump to another one, and so on, covering in the end (but only after a very long time) uniformly the chaotic part of the phase space. These partial barriers are associated with the existence of time scales, which clearly may constrain the spectral statistics. For these systems, Gaussian Ensemble statistics are recovered only if each of these time scales are significantly shorter than the Heisenberg time \( t_H \) defined as \( \hbar / \Delta \) (\( \Delta \) is the mean level spacing) [14, 16]. The non-generic (and thus non-RMT) character of an ergodic system can thus be more precisely associated with the existence of time scale longer than \( t_H \).

More generally, a RMT description thus appears as a “null hypothesis” (absence of relevant information concerning the quantum system), and what should be understood is what are the “relevant” informations, and how quickly they bring the system away from RMT.

## 4 Applications and conclusion

To finish this short review, we shall make a brief tour of some of the physical systems for which the BGS conjecture has been used with profit. Our goal here is not to go into any details into neither the description of these systems nor the results obtained from the RMT description, as most of them could justify a scholarpedia article on their own right. Rather, we shall very briefly give the context and point to some references as an entry point for the interested reader.

- **Atomic spectra**: the transition to chaos and its relation to random matrices have been studied in details for the spectra of the hydrogen atom subjected to a magnetic field [58, 20, 22] as well as for the helium atom [59].

- **Microwave billiards** have also been studied extensively in relationship with the BGS conjecture [54, 23, 2, 37].

- **Mesoscopic physics**: ballistic quantum dots can often be modeled as chaotic quantum billiards, implying that their eigenlevel and eigenstates can be assumed to follow a random matrix description. This random matrix approach has been used to study the transport properties of these ballistic dots in various regimes: open dots [6, 29]; Coulomb blockade (fluctuations of peak heights [30, 21] or peak spacing [52, 56, 1, 31, 57, 55]); or superconducting dots [26]. It has also formed the basis of the description of the mesoscopic Kondo problem [33, 32, 34].

- **Linear acoustics** in solids [41] and oceans [25].

- **Decoherence** [5], fidelity and Lochschmit echo [28, 27, 19].

- **Riemann zeta function** and generalized L-functions [9, 36, 35].

- **Other applications** includes nuclear physics [13], quantum information [45], and extreme statistics [38].

As the variety of these examples demonstrate what makes the strength of the BGS conjecture is that it states a universal property of chaotic system, and therefore can apply to a wide range of physical phenomena. It belongs now to the toolbox of modern theoretical physics.
A The chaos hierarchy

In the appendix, we consider a motion in the phase space \( \mathcal{P} = \{x = (q, p)\} \) governed by the conservative Hamiltonian \( H(q, p, f) \).

We note
- \( g^t : \mathcal{P} \rightarrow \mathcal{P} \) the Hamiltonian flow, which maps any point \( x_0 = (q_0, p_0) \) in phase space to \( x(t) = g^t x_0 \), the position at time \( t \) of a trajectory initiated at time \( t = 0 \) at \( x_0 \).
- \( \sigma_E \) the normalized projection of the Liouville measure on the energy surface \( E \):

\[
d\sigma_E = \frac{\delta(E - H)dpdq}{\int_p \delta(E - H)dpdq}
\]

A.1 Ergodicity

Definition 1 The classical motion is said to be ergodic on the energy surface \( S_E \) if the only ensembles invariant under the Hamiltonian flow \( g^t \) are either of measure 1 or of measure 0.

i.e. \( \forall \Omega \subset S_E, (g^t \Omega = \Omega \text{ for all } t) \Rightarrow (\sigma_E(\Omega) = 0 \text{ or } 1) \)

NB: ergodicity alone is barely considered as chaos (eg dim 1):

![Figure 4](image)

Figure 4: For a one degree of freedom (and thus integrable) conservative system, the motion is obviously ergodic on \( S_E \).

Definition 2 The motion is ergodic if for any (integrable under \( \sigma_E \)) function \( f \) and for almost all \( x_0 \) in \( S_E \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(g^t x_0) = \int_{S_E} f(x) d\sigma_E .
\]

Definitions 1 and 2 are equivalent.

A.2 Mixing

Definition 1 The motion on \( S_E \) is said to be “mixing” if for any two (non-zero measure) parts \( A \) and \( B \) of \( S_E \)

\[
\lim_{t \to \infty} \sigma_E(B \cap g^t A) = \sigma_E(B) \cdot \sigma_E(A)
\]

Remark: Mixing implies ergodicity.
A.3 K Systems (Kolmogorov)

Intuitively, chaotic systems are associated with the notion of exponential separation of trajectories (which is actually the point of view taken for “Axiome A” systems). Kolmogorov proposed instead to characterize chaos through the notion of information.

Let us consider here a dynamical system \((M, \mu, T)\), where \(M\) is the space on which the motion takes place, \(\mu\) is a measure on this space, and \(T\) is a bijective map on \(M\). The Hamiltonian motion considered in this appendix enters into this general framework provided one discretizes the time \((t = t_0, 2t_0, 3t_0, \cdots)\), in which case we can take \(M \equiv \mathcal{E}, \mu \equiv \sigma_{\mathcal{E}}\) and \(T \equiv g_{t_0}\).

Let us consider a partition \(A = \{A_i\}\) of the space \(M\) \((M = \bigcup A_i, (A_i \cap A_j = \emptyset \forall i \neq j))\). The question which we ask ourselves is: “assuming we know the list of all the cells \(A_i(t)\) visited in the past, i.e. \(\{i(t); t = -\infty, \cdots, -3t_0, -2t_0, -t_0, 0\}\), do we gain any new information by learning where will be the trajectory at time \(t = t_0\)?”

More formally:

**Definition 2** Let \(A\) a partition of \(M\). The entropy \(H\) of \((A)\) is

\[
H(A) = -\sum_i \sigma_{\mathcal{E}}(A_i) \ln[\sigma_{\mathcal{E}}(A_i)]
\]

(i.e. \(H(A)\) is the average information obtained from the knowledge of the cell \(A_i\) to which belongs a point taken at random in \(M\)).

**Definition 3** The entropy of \(T\) relative to the partition \(A\) is

\[
h(T, A) \equiv \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{k=0}^{n-1} T^{-k} A)
\]

\((A \bigvee B \equiv \{A_i \cap B_j\})\).

One can show that \(h(T, A) = H(A/ \bigvee_{k=-\infty}^{1} T^{k} A)\), where

\[
H(A/B) = -\sum_{i,j} \mu(A_i \cap B_j) \ln \left( \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \right)
\]

(The entropy of “\(A\) knowing \(B\)” is the quantity of information acquired on average by learning in which cell of the partition \(A\) is a point taken at random in \(M\) but assuming known the cell of the partition \(B\) to which it belongs.)

Stating that \(h(T, A) = 0\) means that once the entire past of a trajectory is known, the addition of an extra time step does not bring any new information.
**Definition 4** A dynamical system $(M, \mu, T)$ is a K-system if for any non-trivial partition $\mathcal{A}$ of $M$, $h(T, \mathcal{A}) > 0$.

*(NB: for continuous time systems, this property of course does not depend on the choice of the time step $t_0$)*

**Remark:** K-systems are necessarily mixing, and thus ergodic.

**Definition 5** The $\sup\{h(T, \mathcal{A})\}$ is the Kolmogorov-Sinai entropy of the system (for continuous time system, we then assume $t_0 = 1$).

### A.4 B-Systems (Bernoulli)

**Definition 6** A partition $\mathcal{P}$ of $M$ is said to be a generating partition if $\bigvee_{k=-\infty}^{+\infty} T^k \mathcal{P}$ is the partition into point of $M$.

A generating partition is therefore such that the complete list of cells visited in the past and in the future completely specifies a trajectory.

**Definition 7** A dynamical system $(M, \mu, T)$ is Bernoulli if one can find a generating (for $T$) partition $\mathcal{P}$ such that for all $j$, $\bigvee_{k=1}^{j+1} T^k \mathcal{P}$ and $T^j \mathcal{P}$ are independent.

Saying that two partitions $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_i\}$ are independent implies that $\forall (i, j)$, $\mu(A_i \cap B_j) = \mu(A_i) \mu(B_j)$. The independence of $\bigvee_{k=1}^{j-1} T^k \mathcal{P}$ and $T^j \mathcal{P}$ thus implies that the knowledge of all the cells visited during $j$ successive iterations provides no indication concerning which cell will be visited at the next iteration.

**Remark:** All B-systems are K-systems.

### A.5 A-Systems (Anosov)

In this subsection, the map $T$ is assumed to be a diffeomorphism and $M$ a $C^\infty$ compact Riemannian manifold.

![Figure 6](image-url)

**Definition 8** A (closed) subset $\Lambda \subset M$ is hyperbolic if for any point $x \in \Lambda$ the tangent space $T_x M$ can be written as a direct sum $T_x M = E^u_x \oplus E^s_x$ such that

"Figure 6:"

"Definition 8" A (closed) subset $\Lambda \subset M$ is hyperbolic if for any point $x \in \Lambda$ the tangent space $T_x M$ can be written as a direct sum $T_x M = E^u_x \oplus E^s_x$ such that"
i) \[
DT(E^s_x) = E^s_{T(x)} \quad DT(E^u_x) = E^u_{T(x)}
\]

ii) there is a constant \( c > 0 \) and a constant \( \lambda \in ]0,1[ \) such that
\[
DT^n(v) \leq c\lambda^n \|v\| \quad \text{for} \ v \in E^s_x \ (n \geq 0)
\]
\[
DT^{-n}(v) \leq c\lambda^n \|v\| \quad \text{for} \ v \in E^u_x \ (n \geq 0).
\]

ii) \( E^s_x \) and \( E^u_x \) vary continuously with \( x \).

“Axiom A system” (Anosov) are essentially hyperbolic systems (in the sense defined above) for which there is no complication coming from trajectories wandering to infinity.

**Definition 9** \( x \in M \) is non-wandering if for any neighborhood \( U \) of \( x \)
\[
U \cap \bigcup_{n \geq 0} T^n U \neq \emptyset
\]

**Definition 10** Let \( \Omega(T) = \{x/x \text{ non-wandering}\} \), \( T \) is said to be axiom A if \( \Omega(T) \) is hyperbolic and if
\[
\Omega(T) = \{x/x \text{ periodic}\}.
\]

Axiom A systems are Bernoulli.

**Stable and unstable manifold**

Let
\[
W^s(x) = \{y \in M/d(T^n x, T^n y) \to 0 \text{when} \ n \to \infty\}
\]
\[
W^*_s(x) = \{y \in M/d(T^n x, T^n y) \leq \epsilon \ \forall n \geq 0\}
\]

and
\[
W^u(x) = \{y \in M/d(T^{-n} x, T^{-n} y) \to 0 \text{when} \ n \to \infty\}
\]
\[
W^*_u(x) = \{y \in M/d(T^{-n} x, T^{-n} y) \leq \epsilon \ \forall n \geq 0\}
\]
If \( T \) is Axiom A, then
- **exponential convergence**

\[
d(T^n x, T^n y) \leq \lambda^n d(x, y) \quad \text{for} \ y \in W^*_s(x) \ n \geq 0
\]
\[
d(T^{-n} x, T^{-n} y) \leq \lambda^n d(x, y) \quad \text{for} \ y \in W^*_u(x) \ n \geq 0
\]

(in particular \( (W^*_u(x) \subset W^u(x)) \) and \( (W^*_s(x) \subset W^s(x)) \)).
- For all \( \epsilon > 0 \), on can find \( \delta > 0 \) such that \( W^*_s(x) \cap W^u(\epsilon) \) is a single point if \( d(x, y) \leq \delta \) (and, of course \( x, y \in \Omega(T) \)).
B  Spectral statistics

In this appendix, we introduce a few spectral statistics often encountered in the context of random matrix theory, together with some of the corresponding expressions for the Wigner Gaussian ensembles.

B.1 Unfolding the spectra

Consider a quantum spectrum \( \epsilon_0, \epsilon_1, \epsilon_2, \cdots \) which either corresponds to the energy levels of a given quantum systems or is a random sequence generated by diagonalizing a realization within some ensemble of matrix. Let \( d(\epsilon) = \sum_n \delta(\epsilon - \epsilon_n) \) the density of state.

In the random case, the mean density of states \( \langle d(\epsilon) \rangle \) is naturally defined by averaging \( d(\epsilon) \) over the ensemble. For a non-random energy sequence corresponding to a specified quantum system, the mean density of state can still be defined, but this time through an energy average

\[
\langle d(\epsilon) \rangle \equiv \frac{1}{\Delta \epsilon} \int_{\epsilon - \Delta \epsilon / 2}^{\epsilon + \Delta \epsilon / 2} d(\epsilon') d\epsilon'.
\]  

This definition assumes of course that one can find a range of \( \Delta \epsilon \) large enough to average the fluctuations of \( d(\epsilon) \), but small enough to be negligible on the scale at which \( \langle d(\epsilon) \rangle \) itself varies, in such a way that the latter is independent of \( \Delta \epsilon \). Such a range of \( \Delta \epsilon \) generically exists in the semiclassical limit. In the following the average \( \langle \cdot \rangle \) will mean either the ensemble average or the energy average Eq. (8) depending on whether one considers an ensemble of matrices or one specific physical system.

The mean density of states \( \langle d(\epsilon) \rangle \) is usually governed by the phase space volume of the classical energy surface \( \epsilon \), and is therefore not related to the nature, chaotic or integrable, of the classical dynamics. To isolate the spectral fluctuations from the mean behavior of the density, it is customary to unfold the spectra, that is to map the sequence \( \epsilon_0, \epsilon_1, \epsilon_2, \cdots \) into a new sequence \( x_0, x_1, x_2, \cdots \) with the same fluctuation properties but with a mean density equal to one.

Introducing the counting function \( N(\epsilon) \equiv \int^\epsilon d(\epsilon') d\epsilon' \), and \( \langle N(\epsilon) \rangle \) is average value, it can be checked easily that the transformation

\[
x_j = \langle N(\epsilon_j) \rangle
\]
performs the required unfolding. In the rest of this appendix we shall consider only the unfolded sequence \(x_0, x_1, x_2, \ldots\). We note
\[
    d(y) \overset{\text{def}}{=} \sum_n \delta(y - x_n)
\]
the density of unfolded states.

### B.2 \(n\)-point correlation functions and Dyson cluster functions

The simplest spectral statistics are the \(n\)-point correlation functions
\[
    R_n(y_1, y_2, \ldots, y_n) \overset{\text{def}}{=} \langle d(y_1) d(y_2) \cdots d(y_n) \rangle.
\]
\(R_1(y)\) is just the mean density of unfolded level \(\langle d(y) \rangle\), which, by construction, is equal to one.

If the energy levels are uncorrelated, the \(n\)-point correlation function reduces to
\[
    R_n(y_1, y_2, \ldots, y_n) = R_1(y_1) R_1(y_2) \cdots R_1(y_n) \quad (= 1 \text{ here}).
\]
To focus on the correlations, Dyson has introduced the cluster functions
\[
    Y_n(y_1, y_2, \ldots, y_n) \overset{\text{def}}{=} \sum_G (-1)^{n-m} (n-1)! \prod_{j=1}^m R_{G_j}(y_t; t \in G_j),
\]
where \(G = \) all partitions of the set of indices \(\{1, 2, \ldots, n\}\) in \(m\) sub-parts \(\{\{1, 2, \ldots, n\}\} = \bigcup_{j=1}^m G_j\).

For instance
\[
    Y_2(y_1, y_2) = R_2(y_1, y_2) - R_1(y_1) R_1(y_2),
\]
\[
    Y_3(y_1, y_2, y_3) = R_3(y_1, y_2, y_3) - R_1(y_1) R_2(y_2, y_3) - R_1(y_2) R_2(y_3, y_1) - R_1(y_3) R_2(y_1, y_2) + 2 R_1(y_1) R_1(y_2) R_1(y_3).
\]

For systems invariant by (energy) translation, one notes
\[
    R_2(s) \overset{\text{def}}{=} R_2(y - s/2, y + s/2) = \langle d(y - s/2) d(y + s/2) \rangle,
\]
\[
    Y_2(s) = R_2(s) - 1
\]
and one can introduce the form factor \(K(\tau)\) which is just the Fourier transform of \(Y_2(s)\)
\[
    K(\tau) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} Y_2(s) e^{2i\pi\tau s} ds.
\]

For the Gaussian Orthogonal and Unitary Ensembles, the two-point cluster function can be expressed as
\[
    \text{(GOE):} \quad Y_2(y) = \frac{\sin^2 y}{y^2} - (\text{Si}(\pi y) - \pi \epsilon(y)) \left( \frac{\cos \pi y}{\pi y} - \frac{\sin \pi y}{(\pi y)^2} \right)
\]
\[
    \text{(GUE):} \quad Y_2(y) = \left( \frac{\sin \pi y}{\pi y} \right)^2.
\]
\(\text{Si}(y) = \int_0^y \frac{\sin x}{x} dx, \quad \epsilon(y) = 1/2 \text{ for } y > 0 \text{ and } -1/2 \text{ for } y < 0, \text{ and } \gamma \simeq 0.5772 \cdots \text{ is the Euler constant.}\) For uncorrelated levels, one has essentially by definition \(Y_2(y) \equiv 0\).

Taking the Fourier transform leads in the GUE case to
\[
    K(\tau) = 1 - \tau \quad (0 < \tau < 1)
\]
\[
    = 0 \quad (\tau > 1).
\]
The expression of \(K(\tau)\) for GOE is more involved, but starts as \(K(\tau) = 1 - 2\tau + O(\tau^2)\) at small \(\tau\). The factor two between the slopes at the origin is actually quite profound, and is related to the existence of time reversal symmetric periodic orbits for the time reversal invariant systems for which GOE apply.
B.2.1 Nearest neighbor spacing

Let \( x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \) the rescaled energy levels ranked by increasing order, and \( s_j = x_{j+1} - x_j \) the spacing between the \( j^{th} \) and \((j+1)^{th}\) levels. The nearest neighbor spacing distribution is defined as

\[
P_{\text{nn}}(s) = \langle \frac{1}{N} \sum_{j=1}^{N} \delta(s - s_j) \rangle.
\]

Note that having a spacing \( s \) implies that for some energy \( y \), a pair of levels is located in \( y \) and \( y + s \), but also that no other level lies in the range \([y, y + s]\). As a consequence \( P(s) \) actually mixes informations about \( n \)-point correlation functions of all orders.

For uncorrelated levels, the nearest neighbor distribution is given by Poisson:

\[
P_{\text{nn}}(s) = \exp(-s).
\]

For the Gaussian ensembles, a very precise approximation of the \( N \to \infty \) result is provided by the result of the “Wigner surmise”, which consists in computing

\[
P_{\text{nn}}(s) = \begin{cases} 
\pi^2 / 2 & \text{for GOE (\( \beta = 1 \))} \quad (18) \\
32 / \pi^2 s^2 & \text{for GUE (\( \beta = 2 \))} \quad (19) \\
2 / 3^2 \pi^4 s^4 & \text{for GSE (\( \beta = 4 \))} \quad (20) 
\end{cases}
\]

\( P_{\text{nn}}(s) \) focuses on short range correlations (\( s \lesssim 1 \), which correspond to spacing smaller than the mean level spacing for the original energies), which reveals the level repulsion characteristic of the Wigner ensembles.

B.2.2 Number variance

Let

\[
n(L) = \int_{\tilde{y}}^{\tilde{y}+L} d(y)dy
\]

the number of rescaled energy levels contained in an interval of length \( L \). Because the mean density of rescaled energy is fixed to one by construction, \( \langle n(L) \rangle = L \). However its variance

\[
\Sigma(L) \overset{\text{def}}{=} \langle n(L)^2 \rangle - L^2 = L - 2 \int_0^L (L - s)Y_2(s)ds
\]

contains informations about correlations at long distance (\( s \gg 1 \), which correspond to spacing larger than the mean level spacing for the original energies).

For the Gaussian Orthogonal and Unitary Ensembles, the number variance can be expressed as

\[
\begin{align*}
\Sigma_{\text{GUE}}^2(L) &= \frac{1}{\pi^2} \left[ \log(2\pi L) + \gamma + 1 - \cos(2\pi L) \right. \\
&\quad \left. - \text{Ci}(2\pi L) \right] + L \left[ 1 - \frac{2}{\pi} \text{Si}(2\pi L) \right] + O(L^{-1}) \quad (\beta = 1 \text{ for GOE, and } \beta = 2 \text{ for GUE}).
\end{align*}
\]

The special functions \( \text{Si}(y) = \int_0^y \frac{\sin x}{x} dx \) and \( \text{Ci}(y) = \gamma + \log y + \int_0^y \frac{\cos x}{x} dx \).
This logarithmic dependence for large $L$ should be contrasted with the linear dependence
$\Sigma_{\text{Poisson}}^2(L) = L$ valid for uncorrelated levels. For instance, an interval of length $L = 100$
will contain anywhere from 90 to 110 levels in the Poisson case, but exactly 100 levels plus or
minus one level in the Gaussian case.

B.2.3 Dyson-Mehta function

Consider again the interval $[\tilde{y}, \tilde{y} + L]$, and the “step function"

$$N(y) \overset{\text{def}}{=} \sum_{x_i \in [\tilde{y}, \tilde{y} + L]} \Theta(y - x_i).$$

Define then

$$\Delta(L) \overset{\text{def}}{=} \min_{a,b} \int_{\tilde{y}}^{\tilde{y}+L} [N(y) - (ay + b)]^2 dy.$$

The Dyson-Mehta function is then defined as the mean value

$$\bar{\Delta}(L) = \langle \Delta(L) \rangle.$$ (23)

In an experimental context in which some of the levels might be missed, the statistics $\bar{\Delta}(L)$
show less sensitivity to these experimental uncertainties than other spectral statistics, and have
been introduced for this purpose by Mehta and Dyson to analyze nuclear spectra.

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