An [imaginary time] Schrödinger approach to mean field games

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Mean Field Games (MFG) provide a theoretical frame to model socio-economic systems. In this letter, we study a particular class of MFG which shows strong analogies with the *non-linear Schrödinger and Gross-Pitaevskii equations* introduced in physics to describe a variety of physical phenomena. Using this bridge many results and techniques developed along the years in the latter context can be transferred to the former, which provides both a new domain of application for the non-linear Schrödinger equation and a new and fruitful approach in the study of mean field games. Utilizing this approach, we analyze in details a population dynamics model in which the "players" are under a strong incentive to coordinate themselves.

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Mean field games, were introduced a decade ago by J-M.Lasry and P-L. Lions [1, 2] and by M. Huang and co-workers [3] as a tractable version of game theory for a large number of players. This approach provides a very versatile framework to model a vast range of socio-economic problems ranging from social behavior [4–7] to finance and economy [8–10]. Phrased in the language of macroeconomy, it makes it possible to go beyond the "representative agent" description [10] and introduce, through its game-theory component, some of the complexity associated with the variability of economic agents' situations. It does so while keeping some reasonable degree of simplicity thanks to the "mean-field" point of view taken. In engineering science, it also proposes a manageable framework to approach complex optimization problems involving a large number of coupled subsystems [3, 11].

This relatively new field has witnessed a very rapid development in the last few years, and has followed two major avenues. The first one is a mathematical approach in which one aims at proving the internal consistency of the theory [12–14] as well as deriving other rigorous results such as existence and uniqueness of solutions for some classes of models [15, 16]. The other direction taken was to develop efficient numerical schemes [5, 17, 18]. One thing which has, however, prevented the diffusion of this tool at a significantly larger scale is the lack of effective approximation schemes. In fact, in spite of the "mean-field-type" assumptions, the constitutive equations of these models remain rather difficult to analyze, in particular because of their atypical forward-backward structure, and only a few simple models admit an analytical solution [6, 19–21]. On the other hand, full fledged numerical analyses of the mean field games equations leave much to be understood.

We show here that there is a strong and deep relationship between mean field games (or at least a large class of them), and the non-linear Schrödinger (or Gross-Pitaevskii) equation, which has been studied for almost a century by physicists to describe various physical systems ranging from interacting bosons in the mean field approximation to gravity waves in inviscid fluids. The goal of this paper is to show that this identification allows to transfer to mean field games (or at least to a class of them) a vast array of knowledge and techniques that have been developed through the years in this field (see e.g. [22–26]). In particular, this opens the way to very effective approximation schemes leading both to a qualitative understanding and a good quantitative description of the solutions of the mean field games equations. This applies to many circumstances where a direct analysis of the mean field games equations seems highly non-trivial, and in any case has not been fully undertaken. In particular we show how this approach provides an essentially complete description of the regime of strong, short range, attractive interactions, which is presumably the most interesting case.

From a formal point a view, a mean field game is defined by two components: the motion of the agents and the quantity they try to optimize. Each agent $i = 1, \cdots N$ is assumed to be characterized by a "state variable" $X_i(t) \in \mathbb{R}^n$, which, depending on the problem under consideration, may represent physical space [5], the amounts of some natural resources [9], or the position of a portfolio [8]. The dynamics of X_i contains a deterministic part which is controlled by the agent, and a random one associated with external noise. The simplest form of such a motion is a Langevin dynamics

$$dX_i = a_i(t)dt + \sigma dW_i \ , \tag{1}$$

where W_i is a white noise of variance one. On the other hand, each agent chooses the drift $a_i(t)$ at time t in order to minimize a cost function whose typical form is:

$$\begin{aligned} c[a_i](X_i(t),t) &= \langle\!\langle c_T(X_i(T)) \rangle\!\rangle_{\text{noise}} \\ &+ \langle\!\langle \int_t^T \left(\frac{\mu}{2} a_i^2(\tau) - V[m_\tau](X_i(\tau)) \right) d\tau \rangle\!\rangle_{\text{noise}} \ . \end{aligned} \tag{2}$$

In this equation, $\langle\!\langle \cdot \rangle\!\rangle_{\text{noise}}$ means an average over the noise, $\mu > 0$ tunes the cost of a high drift velocity, $c_T(x)$ is the final cost paid at the end of the optimization period T, and $V[m_t](x)$ is both a function of x and a functional of the density of agents $m_t(x) \equiv \frac{1}{N}\sum_i \delta(x - X_i(t))$.

The class of models defined by Eqs. (1-2), which includes the "population dynamics models" introduced by Guéant and co-workers [9], can be thought of as an equivalent of the "Ising model" in the context of mean field games: it is representative of most mean field games while keeping clear of rather serious "technicalities" attached to specific, application oriented, models. These more realistic mean field games may require to consider other forms of cost function or dynamics [15, 27], as well as the introduction of inhomogeneities in the agents characteristics [8]. At the present stage, very little is known about the solutions of the mean field games equations even in their "Ising-model" like form Eqs. (1-2), to which we therefore limit our discussion.

Defining the value function $u(x,t) \equiv \min_{a_i(.)} c[a_i](x,t)$, the minimization of the cost function Eq. (2), under the dynamics Eq. (1), leads to a system of coupled partial differential equations [2]:

$$\partial_t u - \frac{1}{2\mu}(\partial_x u)^2 + \frac{\sigma^2}{2}\partial^2_{xx} u = V[m_t](x) \ , \tag{3}$$

$$\partial_t m + \partial_x(a(x,t)m) - \frac{\sigma^2}{2}\partial^2_{xx} m = 0 \ , \tag{4}$$

with $a(x,t) \equiv -\frac{1}{\mu}\partial_x u(x,t)$. Eq. (3) is a Hamilton-Jacobi-Bellman (HJB) equation propagating the value function $u(x,t)$ backward in time from the final condition $u(x,T) \equiv c_T(x)$; Eq. (4) is a Fokker-Planck (FP) equation propagating the density of agent $m_t(x) = m(x,t)$ forward in time from the initial condition $m_0(x)$. The two equations (3) and (4) are coupled due to the density dependence of the "potential" $V[m_t](x)$ and by the fact that the optimized drift $a(x,t)$ is the gradient of the value function.

With a relatively simple change of variables [28], the system, Eqs. (3-4), can be cast in a form which we identify here as an *imaginary time* version of the *non-linear Schrödinger equation*. As a consequence of this identification, we show hereafter that the associated formalism can be naturally introduced, leading to an effective approximation scheme. In particular, this approach relates to a very deep theorem derived by Cardialaguet and coworkers [29] which states that (under additional technical conditions) there exists an *ergodic state* $m^*(x)$ in the long time limit that the density $m(x,t)$ approaches for T large when the time t is sufficiently far from both 0 and T.

To proceed, we introduce two new functions: $\Phi(x,t) = \exp[-u(x,t)/\mu\sigma^2]$ (which corresponds to a Cole-Hopf transformation for the HJB equation), and $\Gamma(x,t) = m(x,t)/\Phi(x,t)$. Eqs. (3-4) then read for these new variables:

$$-\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\partial^2_{xx}\Phi + V[m_t](x)\Phi \ , \tag{5}$$

$$\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\partial^2_{xx}\Gamma + V[m_t](x)\Gamma \ , \tag{6}$$

with the final condition $\Phi_T(x) \equiv \Phi(x,T) = \exp[-u_T(x)/\mu\sigma^2]$ and the initial condition $\Gamma(x,0)\Phi(x,0) = m_0(x)$.

Under the formal replacement $\mu\sigma^2 \to -i\hbar$, these equations are exactly those governing the evolution of a wavefunction and its complex conjugate under the quantum Hamiltonian $\hat{H} = \hat{\Pi}^2/(2\mu) + V[m_t](\hat{X})$, where $\hat{\Pi} \equiv \mu\sigma^2\partial_x$ and $\hat{X}$ are respectively momentum and position operators.

For an arbitrary operator $\hat{O} = f(\hat{X},\hat{\Pi})$, let us introduce the average

$$\langle\hat{O}\rangle(t) \equiv \langle\Gamma(t)|\hat{O}|\Phi(t)\rangle = \int dx\,\Gamma(x,t)\hat{O}\Phi(x,t) \ ,$$

which, whenever $\hat{O} = O(\hat{X})$, reduces to the classical mean value $\int dx\, m(x,t)O(x)$. One has, as for the Schrödinger equation, $\mu\sigma^2 \frac{d}{dt}\langle\hat{O}\rangle = \langle[\hat{H},\hat{O}]\rangle$. In particular, straightforward algebra gives

$$\frac{d}{dt}\langle\hat{X}\rangle = \frac{\langle\hat{\Pi}\rangle}{\mu} \ , \qquad \frac{d}{dt}\langle\hat{\Pi}\rangle = \langle\hat{F}\rangle \ , \tag{7}$$

where we have introduced the "force" operator $\hat{F}[m_t] \equiv -\partial_x V[m_t](\hat{X})$. The variance $\Sigma^2(t) \equiv \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$ evolves according to:

$$\frac{d}{dt}\Sigma^2 = \frac{1}{\mu}\left(\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle\right) \ , \tag{8}$$

$$\frac{d^2}{dt^2}\Sigma^2 = \frac{2}{\mu^2}\left(\langle\hat{\Pi}^2\rangle - \langle\hat{\Pi}\rangle^2\right) - \frac{2}{\mu}\left(\langle\hat{X}\hat{F}\rangle) - \langle\hat{X}\rangle\langle\hat{F}\rangle\right) \ . \tag{9}$$

If furthermore one considers potentials of the form [27]

$$V[m_t](x) = U_0(x) + g\ m_t(x)^\alpha \ , \tag{10}$$

with $\alpha > 0$, one gets explicitly

$$\langle\hat{F}\rangle = \langle\hat{F}_0\rangle \equiv \langle-\nabla_x U_0(\hat{X})\rangle \ , \tag{11}$$

$$\langle\hat{X}\hat{F}\rangle = \langle\hat{X}F_0\rangle - \alpha\langle H_{\rm int}\rangle \ , \tag{12}$$

with $\langle H_{\rm int}\rangle \equiv (g/(\alpha+1))\int dx\ m_t^{\alpha+1}(x)$; moreover the "total energy"

$$\mathcal{E}(t) \equiv \frac{1}{2\mu}\langle\hat{\Pi}^2\rangle + \langle U_0(\hat{X})\rangle + \langle H_{\rm int}\rangle \tag{13}$$

is a conserved quantity, i.e. $d\mathcal{E}/dt \equiv 0$.

Our claim is that Eqs. (7-13), together with many results known in the context of the non-linear Schrödinger equation, can form the basis of the analysis of a very large class of mean field games for various associated potentials, including some long range interactions. In the following, we will illustrate our point of view, restricting ourselves to the *one dimensional case* and to potentials of the form Eq. (10) (though most of our findings can be extended straightforwardly to other cases). We will furthermore focus mainly on the regime that we think is the most interesting, namely the one of strong positive interactions (g positive and large, in a sense clarified below).

To begin our analysis, it is presumably useful to start with persistent solutions of Eqs. (5-6), which will eventually correspond to the "ergodic state"of Cardialaguet et al. [29]. These are obtained as $\Gamma(x,t) = \psi^*(x)e^{\epsilon t/\mu\sigma^2}$ and $\Phi(x,t) = \psi^*(x)e^{-\epsilon t/\mu\sigma^2}$, giving $m(x,t) = m^*(x) = (\psi^*(x))^2$, where $\psi^*(x)$ is the solution of the time independent non-linear equation $\hat{H}\psi^*(x) = \epsilon\psi^*(x)$, that is

$$\frac{\mu\sigma^4}{2}\partial^2_{xx}\psi^* + U_0(x)\psi^* + g(\psi^*)^{2\alpha+1} = \epsilon\psi^* \ . \tag{14}$$

We specialize from now on to $\alpha = 1$ (the general case $\alpha > 0$ can be addressed following closely the approach described below [30]). In this case Eq. (14) is exactly the (time-independent) Gross-Pitaevskii equation. In the limit $U_0(x) = 0$ the lowest energy state is a soliton [26]:

$$\psi^*_{\rm s}(x) = \frac{1}{\sqrt{2\eta}}\cosh^{-1}\left(\frac{x}{\eta}\right) \ , \tag{15}$$

with $\eta \equiv 2\mu\sigma^4/g$, and $\epsilon_{\rm s} = g/(4\eta)$.

Note that Eq. (15) provides a length scale, η, the spatial extension of the soliton. We now consider a non zero external confining potential $U_0(x)$; by definition of a strong interaction regime, the variations of $U_0(x)$ on a scale η are small, that is $|\eta\nabla_x U_0| \ll |\epsilon_{\rm s}|$ and $|\eta\nabla^2_x U_0| \ll |\nabla_x U_0|$. Under these conditions, it is clear that, away from $t = 0$ and $t = T$ where the boundary conditions may force the density of agents out of the soliton form, $m(x,t)$ will keep a form close to $[\psi^*(x-\bar{x}(t))]^2$, centered around its mean value $\bar{x}(t) \equiv \langle\hat{X}\rangle(t)$. For this narrow density profile one has $\langle\hat{F}_0\rangle \simeq -\nabla_x U_0(\bar{x})$, and applying Eq. (7) readily gives

$$\mu\frac{d^2}{dt^2}\bar{x}(t) = -\nabla_x U_0(\bar{x}(t)) \ . \tag{16}$$

In the strong interaction regime, the motion of the soliton is simply that of a classical particle of mass μ in the potential $U_0(x)$.

The next point we need to address is the formation/destruction of the soliton. Indeed, considering for instance the neighborhood of $t=0$, the initial condition $m_0(x)$ can be taken far from the soliton form, and one may ask how $m(x,t)$ evolves to it from $m_0(x)$. The short answer to this question is: "quickly" – indeed this process is dominated by interactions which are assumed to be large. To obtain further insight, let us assume that the density has initially a Gaussian shape of variance Σ_i^2 and centered around $\bar{x}$. We use a Gaussian ansatz to describe its initial evolution

$$m(x,t) \simeq \frac{1}{\sqrt{2\pi\Sigma^2(t)}}\exp\left[-\frac{(x-\bar{x})^2}{2\Sigma^2(t)}\right] \ .$$

Neglecting the influence of the external potential during the formation of the soliton in Eqs. (9-12), and using that the total energy Eq. (13) is a conserved quantity, we can express $\langle\hat{\Pi}^2\rangle/2\mu$ in terms of $\langle H_{\rm int}\rangle$ and its large t stationary limit $\langle H_{\rm int}\rangle_*$ and obtain

$$\begin{aligned}\frac{d^2}{dt^2}\Sigma^2 &= \frac{2}{\mu}(\langle H_{\rm int}\rangle_* - \langle H_{\rm int}\rangle)\\ &= \frac{g}{2\mu\sqrt{\pi}}\left(\frac{1}{\Sigma_*} - \frac{1}{\Sigma(t)}\right) \ ,\end{aligned} \tag{17}$$

where $\Sigma_* = \sqrt{\pi}\eta$. Imposing $\Sigma(t=0) = \Sigma_i$, and introducing $z_t = \Sigma(t)/\Sigma_*$, $z_i = \Sigma_i/\Sigma_*$, and $\tau^* \equiv 2\pi\sqrt{\mu\eta^3/g}$, Eq. (17) can be integrated as

$$-(z_t - z_i) - \log\left(\frac{1-z_t}{1-z_i}\right) = \frac{t}{\tau^*} \ . \tag{18}$$

The destruction of the soliton can be tackled similarly, except that the terminal condition imposed on Σ^2 is of the mixed form $\mu{\rm d}\Sigma^2/{\rm d}t(T) + 2(\partial^2_{xx}c_T(\bar{x}(T)))\Sigma^2(T) = \sigma^2$, and thus gives a different expression (not shown) for the solution of Eq. (17). One finds that $\Sigma(T) \simeq \Sigma^*(1+\xi)$ where $\xi \simeq 0.43$ is a number. So the final density $m(x,T)$ has a dispersion which remains of order Σ_*.

Setting aside the precise way the soliton is formed or destroyed near the boundaries $t = 0$ and $t = T$, the important point here is that the characteristic time $\tau^* = \pi\eta\sqrt{\mu/|\epsilon_{\rm s}|}$ which emerges is short, in the sense that η is assumed the smallest length scale of the problem and $\epsilon_{\rm s}$ the largest energy scale of the problem. This is consistent with the fact that during its formation, the soliton can be considered immobile and centered around $\bar{x}$. The terminal condition on the other hand does not involve directly $m(x)$ as what is fixed is the final cost function $c_T(x)$. Using again that near T the density remains localized on a scale $\sim \Sigma_* \sim \eta$ which is short, one can show however that one has for the center of the soliton $\bar{x}(t)$ the terminal condition

$$\mu\frac{d\bar{x}}{dt}(T) = -(\partial_x c_T)[\bar{x}(T)] \ . \tag{19}$$

As an illustration, let us consider a simple population dynamics model in a one dimensional space (say some aquatic species living along a river). Here the state X is the physical position of each individual, the potential $U_0(x)$ represents the intrinsic quality of the location x (for instance for food location) and the mutual interaction $V[m](x)$ the incentive to stay within the group (as a

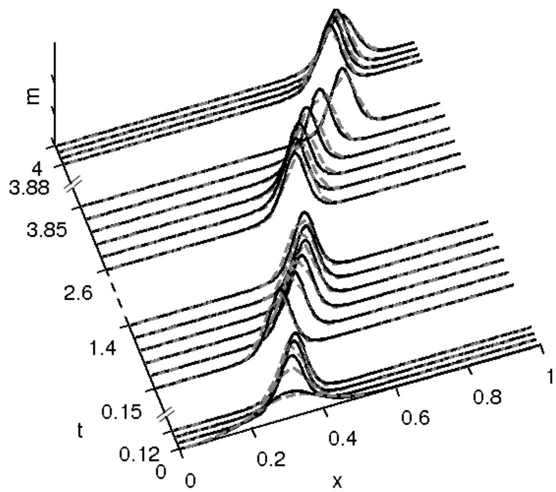

Figure 1. Solution of the mean field game equations for the density of agents in a typical configuration. Solid black: numerical solution of Eqs. (3-4); Dashed grey : Solution for the Gaussian ansatz solution of Eqs. (16-18). The initial density is $m(x,t=0) = \frac{1}{2\eta_i}\cosh^{-2}\left(\frac{x-x_i}{\eta_i}\right)$ with $x_i = 0.3$, $\eta_i = 0.2$, and the final cost $c_T(x) = 2\pi^2(x-0.8)^2$. The parameters of the model are $\sigma = 0.45$, $\mu = 1$, $T = 4$; the potential is as in Eq. (10) with $\alpha = 1$, $g = 2$ and $U_0(x) = -(\pi^2/8)(x-0.5)^2$. To make more visible the creation and relaxation of the soliton the time scale of the initial and final time periods have been magnified.

defense against predators). If the optimization process is running over a day, $m_0(x)$ would be the initial distribution of the group in the morning, and $c_T(x)$ would represent the intrinsic quality of shelter found at the end of the day. Fig. (1) shows a comparison between a numerical solution of Eqs. (3-4) for a potential as in Eq. (10) with $\alpha = 1$ and the predictions derived from the above analysis (see caption for the precise parameters). The quantitative agreement is seen to be very good.

More generally, we can now give a fairly complete description of the solution of this population dynamics model in the regime of strong short-ranged positive interactions that we consider here. One can distinguish three distinct periods of time

In the first one (the formation of the herd, which takes place on the shortest time scale τ^*), the individuals coordinate themselves through their strong mutual interaction and evolve from their initial distribution $m_0(x)$ to a localized one whose extension Σ_* results from a balance between the agents' interaction (which tends to reduce Σ_*), and noise (which tends to increase it). Whenever the Gaussian ansatz is accurate during this phase, Eqs. (17-18) provide a quantitative description of the time evolution of the density of agent. If $m_0(x)$ is not well approximated by a Gaussian this description is presumably a bit more qualitative. Note however that the only place where the Gaussian form has been explicitly used here is when expressing $\langle H_{\rm int}\rangle$ in terms of the variance $\Sigma^2(t)$ of $m(x,t)$. As long as this relation is approximately maintained, and given that $m(x,t)$ has to converge to a soliton form which is well approximated by a Gaussian, the description Eqs. (17-18)) should be reasonably accurate.

The third (and last) time period extends also over the short time scale τ^* just before T, when the population density slightly relax from the soliton form to adjust to the final cost function $c_T(x)$. Since the boundary condition does not involve the final density $m(x,T)$ one can assume there a compact form for $m(x,t)$ with a finite spread on a scale $\sim \Sigma_*$.

In between, assuming of course $T \gg \tau^*$, most of the time period $[0,T]$ is characterized by the relatively slow motion of the population following Eq. (16). Because τ^* is so short, and because the dynamics of $\langle\hat{X}\rangle$ and $\langle\hat{\Pi}\rangle$ are controlled by the external potential $U_0(x)$, their values barely move during the formation or the dispersion of the herd, and thus Eq. (16) can be assumed to be valid all along $[0,T]$. Therefore the precise way in which the herd is initially formed and eventually dispersed will not change drastically what will happen during the propagation phase.

In the intermediate phase, the dynamics is therefore determined: by $m_0(x)$, which fixes the initial position of the herd; by $c_T(x)$, which sets the final velocity of the herd; and by the confining potential $U_0(x)$ which drives the motion between the two. We arrive thus at this relatively non-intuitive result that *the details of the strong coordination between the agents, which is assumed to be the largest force at work here, plays little role in the global picture.*

Considering now the long time limit studied by Cardialaguet and coworker [29], the picture we obtain is the following: the simplest way to form a trajectory fulfilling the boundary condition $\bar{x} = \bar{x}_0$ at $t=0$ and Eq. (19) at $t = T$ for very large T, is to use an initial velocity $\dot{\bar{x}}_0$ such that the energy $E \equiv \mu\dot{\bar{x}}_0^2/2 + U(\bar{x}_0)$ is almost equal to $U_0(x_{\max})$, with $x_{\max}$ the maxima of $U_0(x)$ (which is thus an unstable fixed point). In this way, the trajectory reaches $x_{\max}$ with an almost zero velocity, thus staying there for an arbitrarily long time, before picking speed again to fulfill Eq. (19) at $t=T$. The ergodic state appears in this way as $m_*(x) \equiv [\psi^*(x-x_{\max})]^2$, and is approached exponentially quickly if $U_0(x)$ is at least quadratic around $x_{\max}$.

We stress however that dealing with a boundary condition problem (implying initial and final times) rather than an initial value problem (initial position and velocity fixed) considerably changes things compared to classical mechanics, especially with respect to the uniqueness of the solution. Indeed, if there is more than one local maxima of $U_0(x)$, one can in most circumstances build more than one solution to the problem (depending on the energy $U_0(\bar{x}(t=0))$ and $U_0(\bar{x}(t=T))$, and on the location of the local maxima relative to $\bar{x}(t=0)$ and

$\bar{x}(t = T)$). Taking the solution associated with the lowest value of the cost function Eq. (2) will make it possible to select the correct one, but this process should imply some phase transition: a very small variation of some parameter, for instance the optimization time T, may provoque a discontinuous change and lead the group to explore a completely different area.

In this letter, we have stressed a natural connection between non-linear Schrödinger equations and mean field games expressed by Eqs. (5-6) which makes possible the transfer to this latter field of a large variety of tools to analyze, both qualitatively and quantitatively, a wide class of systems which appear significantly more difficult to address directly in the original form. We have focused on the regime of strong short-ranged interactions but other cases (long range interactions, strong confining potential), and higher dimensional problems, could be addressed very similarly. Exploiting fully this connection provides both a new playground for physicists familiar with the non-linear Schrödinger equation and a path to powerful approximation schemes for mean field games equations. We have illustrated our finding with a stylized population dynamics model, but the analysis of real socio-economic problems should eventually benefit from these progresses.

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