The Lowest Landau Level Anyon Equation of State in the Anti-screening Regime

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Abstract

The thermodynamics of the anyon model projected on the lowest Landau level (LLL) of an external magnetic field is addressed in the anti-screening regime, where the flux tubes carried by the anyons are parallel to the magnetic field. It is claimed that the LLL-anyon equation of state, which is known in the screening regime, can be analytically continued in the statistical parameter across the Fermi point to the antiscreening regime up to the vicinity (whose width tends to zero when the magnetic field becomes infinite) of the Bose point. There, an unphysical discontinuity arises due to the dropping of the non-LLL eigenstates which join the LLL, making the LLL approximation no longer valid. However, taking into account the effect of the non-LLL states at the Bose point would only smoothen the discontinuity and not alter the physics which is captured by the LLL projection: Close to the Bose point, the critical filling factor either goes to infinity (usual bosons) in the screening situation, or to 1/2 in the anti-screening situation, the difference between the flux tubes orientation being relevant even when they carry an infinitesimal fraction of the flux quantum. An exclusion statistics interpretation is adduced, which explains this situation in semiclassical terms. It is further shown how the exact solutions of the 3-anyon problem support this scenario as far as the third cluster coefficient is concerned.

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Identical particles with statistics continuously interpolating between Bose-Einstein and Fermi-Dirac exist in two \[1\] and one \[2\] dimensions. Contrary to the one-dimensional Calogero model, which is solvable, the anyon spectrum is unknown. However, a simplification arises when projecting the anyon model onto the lowest Landau level (LLL) of an external magnetic field, which is justified in the strong field/low temperature limit. A complete linear\(^4\) eigenstate basis, which continuously interpolates between the LLL-bosonic and the LLL-fermionic basis, can be found in the screening regime where the flux \(\phi = \alpha\phi_o\) carried by the anyons is antiparallel to the external magnetic field—more precisely, when the statistics parameter \(\alpha\) which varies from \(\alpha = 0\) (Bose) to \(\alpha = \pm 1\) (Fermi), is such that \(\alpha \in [-1, 0]\) if \(eB > 0\), or equivalently \(\alpha \in [0, 1]\) if \(eB < 0\).

In this situation, the statistical mechanical properties of the anyon gas have been derived \[3\]. Note that in the thermodynamic limit, both the LLL anyon and the Calogero models can be viewed as microscopical realizations of Haldane’s exclusion statistics \[4\] \[5\] \[6\]. Various conformal field theories have also been shown to implement exclusion statistics \[7\].

Clearly, since the magnetic field gives a privileged orientation to the plane, one expects, for a given magnetic field, quite different behavior depending on the sign of \(\alpha\), most particularly in the strong magnetic field limit. Here, one addresses the question of the thermodynamics of the LLL anyon model in the anti-screening regime \(\alpha \in [0, 1]\) if \(eB > 0\) (or \(\alpha \in [-1, 0]\) if \(eB < 0\)), where unknown nonlinear eigenstates should become relevant when their gap above the LLL ground state vanishes in the limit \(\alpha \to 0^+\) (or \(\alpha \to 0^-\)).

Let us start with a short reminder on the \(N\)-anyon model, which is defined in the singular gauge by a free \(N\)-body Pauli Hamiltonian \((\hbar = m = 1)\) \(H_{\text{free}}^u = -2 \sum_{i=1}^N \partial_i \bar{\partial}_i\), \(H_{\text{free}}^d = -2 \sum_{i=1}^N \bar{\partial}_i \partial_i\), where the index \(u, d\) refers here to the spin degree of freedom. The coupling to an external homogeneous magnetic field amounts, in the symmetric gauge, to \(\partial \rightarrow \partial - eBz/4\) and \(\bar{\partial} \rightarrow \bar{\partial} + eBz/4\). The \(N\)-body eigenstates \(\psi_{\text{free}}\) of \(H_{\text{free}}\) have a nontrivial monodromy encoded in the multivalued phase \(\exp(-i\alpha \sum_{k<l} \theta_{kl})\) where \(\sum_{k<l} \theta_{kl}\) is the sum of the angles between pair relative radius vectors and the \(x\) axis in the plane. Looking at the multivalued phase as a singular gauge transformation, one obtains, in the regular gauge, an \(N\)-anyon Aharonov-Bohm Hamiltonian acting on single-valued wave functions (bosonic by convention) with contact interactions \(\mp \pi \alpha \sum_{i<j} \delta^2(z_i - z_j)\) (and \(\mp \sum_i eB/2\) energy shifts) induced by the spin up or spin down coupling to the local magnetic field of the vortices (and to the external magnetic field). The contact interactions have to implement the exclusion of the diagonal of the configuration space, and thus have to be repulsive. So, depending on the sign of \(\alpha\), the spin up Hamiltonian \((\alpha \in [-1, 0])\)

\[4\] Linear refers to the linear dependence in \(\alpha\) of the energy in the presence of a long distance harmonic well regulator.

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or spin down Hamiltonian ($\alpha \in [0, 1]$), is used.

Without loss of generality, let us take in the sequel $eB > 0$, i.e. $\omega_c = +eB/2$, and ignore the trivial $\mp \sum_i eB/2$ Pauli induced shift. Also, in order to compute thermodynamic quantities, a harmonic well of strength $\omega$ is added as a long distance regulator. The thermodynamic limit will always be understood as $\omega \to 0$.

To encode in the eigenstates not only the anyonic multivalued phase but also the short-range repulsion and the long distance Landau and harmonic exponential dampings one sets, if $\alpha \in [-1, 0]$ (screening regime),

$$\psi_{\text{free}} = \prod_{k<l}(z_k - z_l)^{-\alpha}\exp(-\frac{1}{2}\omega_t \sum_{i=1}^N z_i \bar{z}_i)\psi$$

(1)

to obtain an anyonic Hamiltonian acting on $\psi$

$$H = -2 \sum_{i=1}^N \left[ \partial_i \bar{\partial}_i - \frac{\omega_I + \omega_c}{2} z_i \bar{z}_i - \frac{\omega_I - \omega_c}{2} \bar{z}_i \partial_i \right]$$

$$+ 2\alpha \sum_{i<j} \left[ \frac{1}{z_i - z_j} (\bar{\partial}_i - \bar{\partial}_j) - \frac{\omega_I - \omega_c}{2} \right] + \sum_{i=1}^N \omega_t$$

(2)

and, if $\alpha \in [0, 1]$ (anti-screening regime),

$$\psi_{\text{free}} = \prod_{k<l} (\bar{z}_k - \bar{z}_l)^\alpha \exp(-\frac{1}{2}\omega_t \sum_{i=1}^N z_i \bar{z}_i)\psi$$

(3)

so that

$$H = -2 \sum_{i=1}^N \left[ \partial_i \bar{\partial}_i - \frac{\omega_I + \omega_c}{2} z_i \bar{z}_i - \frac{\omega_I - \omega_c}{2} \bar{z}_i \partial_i \right]$$

$$- 2\alpha \sum_{i<j} \left[ \frac{1}{\bar{z}_i - \bar{z}_j} (\partial_i - \partial_j) - \frac{\omega_I + \omega_c}{2} \right] + \sum_{i=1}^N \omega_t$$

(4)

with $\omega_I = \sqrt{\omega_c^2 + \omega^2}$.

Keeping in mind the strong magnetic field limit, one projects (2) and (4) on the LLL basis, that is, on $N$-body eigenstates made of symmetrized (i.e. with bosonic quantum numbers $0 \leq \ell_1 \leq \ldots \leq \ell_N$) products of the 1-body LLL holomorphic eigenstates

$$\left( \frac{\omega_c^{\ell_1+1}}{\pi \ell_1!} \right)^{\ell_1} z_i^{\ell_1}, \quad \ell_i \geq 0$$

(5)

of energy $\omega_c$.

In the screening regime, the LLL projection of the Hamiltonian (2) in the thermodynamic limit gives, trivially, $H = N \omega_c$. This is the LLL anyon model with an infinitely
degenerate $N$-body spectrum. The virtue of the harmonic confinement is to lift the degeneracy with respect to the $\ell_i$’s and to bestow an explicit $\alpha$ dependence on the $N$-body spectrum. In a harmonic well, the LLL eigenstates (3) become

$$
\left( \frac{\omega_{\ell_i} + 1}{\pi \ell_i} \right)^{1/2} z_i^{\ell_i}, \quad \ell_i \geq 0
$$

with a nondegenerate spectrum

$$
\omega_i + (\omega_t - \omega_c)\ell_i, \quad \ell_i \geq 0.
$$

Up to a $\omega_t$ dependent normalization, the LLL anyonic eigenstates in a harmonic well rewrite as

$$
\psi_{\text{free}} = \prod_{i<j} (z_i - z_j)^{-\alpha} \prod_{i=1}^N z_i^{\ell_i} \exp\left(-\frac{1}{2} \omega_t \sum_{i=1}^N z_i z_i^* \right).
$$

Acting on the basis (3), the Hamiltonian (2) narrows down to

$$
H_{\text{LLL}} = N\omega_t + \left[ \sum_{i=1}^N \ell_i - \frac{1}{2} N(N-1)\alpha \right] (\omega_t - \omega_c)
$$

with a harmonic-LLL $N$-anyon spectrum

$$
E_N = N\omega_t + \left[ \sum_{i=1}^N \ell_i - \frac{1}{2} N(N-1)\alpha \right] (\omega_t - \omega_c).
$$

The eigenstates and the spectrum (8,10) interpolate from the harmonic-LLL bosonic to the harmonic-LLL fermionic basis when $\alpha$ goes from 0 to $-1$ and lead, in the thermodynamic limit, to the equation of state

$$
\beta P = \rho_L \ln(1 + \frac{\nu}{1 + \alpha \nu})
$$

and the virial coefficients

$$
a_n = \left( -\frac{1}{\rho_L} \right)^{n-1} \frac{1}{n} \left\{ (1 + \alpha)^n - \alpha^n \right\}.
$$

At the critical filling $\nu_{cr} = -1/\alpha$ where the pressure diverges, one reaches a nondegenerate ground state with all the $\ell_i$’s null: in the singular gauge, it rewrites as

$$
\psi_{\text{free}} = \prod_{i<j} (z_i - z_j)^{-\alpha} \exp\left(-\frac{\omega_t}{2} \sum_{i=1}^N z_i z_i^* \right),
$$

i.e., at the Fermi point $\alpha = -1$, the fermionic Vandermonde determinant built from 1-body Landau eigenstates.
Now the question is: What happens in the anti-screening regime \( \alpha \in [0,1] \)? The relevant Hamiltonian (4) no longer has a simple form when it acts on products of LLL holomorphic eigenstates (6). This translates into the fact that when \( \alpha \to 0^+ \) some, mostly unknown, non-LLL excited eigenstates join the ground state, as indicated by various numerical and semi-classical analyses [9, 10, 11, 12, 13], and as can be appreciated explicitly in the solvable 2-anyon case: The relative 2-anyon spectrum rewrites, when the relative angular momentum \( l \), an even integer, satisfies \( l \geq \alpha \), as

\[
E_{\{n,l\}} = (2n + 1)\omega_t + (l - \alpha)(\omega_t - \omega_c) \to_{\omega \to 0} (2n + 1)\omega_c \tag{14}
\]

and when \( l < \alpha \), as

\[
E_{\{n,l\}} = (2n + 1)\omega_t + (\alpha - l)(\omega_t + \omega_c) \to_{\omega \to 0} (2n + 1)\omega_c + 2(\alpha - l)\omega_c \tag{16}
\]

with the wave functions analytic (anti-analytic) in the relative coordinate \( z = z_1 - z_2 \) for \( l \geq \alpha \) (\( l < \alpha \)).

The bosonic LLL quantum numbers for the 2-body problem are \( n = 0, l \geq 0 \). However, in the presence of the anyonic interaction, the LLL projection happens not to be well defined at the bosonic end \( \alpha \to 0^+ \). Indeed, if \( \alpha \in [-1,0] \), the LLL (analytic) ground state basis obtained from (15) by setting \( n = 0 \) is complete since the \( l = 0 \) state belongs to this basis:

\[
E_{\{n=0,l\geq0\}} = \omega_t + (l - \alpha)(\omega_t - \omega_c) \tag{18}
\]

\[
\psi_{\text{free} \{n=0,l\geq0\}} = z^{l-\alpha} L_n(\omega_t z \bar{z}) \exp\left(-\frac{\omega_t}{2} z \bar{z}\right) \tag{19}
\]

But if \( \alpha \in [0,1] \), this same LLL ground state basis becomes incomplete since the \( l = 0 \) state is now anti-analytic [see (17)] with an energy which varies linearly with \( \alpha \), joining the ground state basis when \( \alpha \to 0^+ \):

\[
E_{\{n=0,l\geq2\}} = \omega_t + (l - \alpha)(\omega_t - \omega_c) \tag{20}
\]

\[
\psi_{\text{free} \{n=0,l\geq2\}} = z^{l-\alpha} \exp\left(-\frac{\omega_t}{2} z \bar{z}\right) \tag{21}
\]

and

\[
E_{\{n=0,l=0\}} = \omega_t + \alpha(\omega_t + \omega_c) \tag{22}
\]

\[
\psi_{\text{free} \{n=0,l=0\}} = \bar{z}^\alpha \exp\left(-\frac{\omega_t}{2} z \bar{z}\right) \tag{23}
\]

It is easy to check that (21,23) are eigenstates of the relative part of the Hamiltonian (4) taking into account the redefinition (3).
Figure 1: The lowest Landau level and the first excited state of the 2-anyon spectrum when $\omega = 0$. 
Figure 2: The second virial coefficient as a function of the statistics parameter $\alpha$: (a) $\frac{a_2}{\lambda^2}$ for zero magnetic field, (b) $\frac{a_2\rho_L}{2}$ for $x = 1$, (c) $\frac{a_2\rho_L}{2}$ for $x = 5$, (d) $\frac{a_2\rho_L}{2}$ for $x \to \infty$.

What is the effect of the anti-analytic eigenstate on the 2-anyon thermodynamics? Consider the second virial coefficient \cite{14}

$$a_2 = \frac{\lambda^2}{x} \left[ -\frac{11 - e^{-2x}}{4} - \frac{1}{2}x - \frac{e^{-2\alpha x} - 1}{(1 + e^{-2x})(1 - e^{-2x})} + (e^{x(|\alpha| - \alpha)} - 1) \right]$$

(24)

where $x = \beta \omega_c$ and the thermal wavelength $\lambda = \sqrt{2\pi \beta}$. In the strong magnetic field limit, $x \to \infty$,

$$a_2 = \frac{1}{2\rho_L} (-1 - 2\alpha) \quad \text{for} \quad \alpha \in [-1, 0] , \quad (25)$$

which indeed agrees with (12), and

$$a_2 = \frac{1}{2\rho_L} \left( -1 - 2\alpha + 4(1 - e^{-2\alpha x}) \right) \quad \text{for} \quad \alpha \in [0, 1] . \quad (26)$$

In considering the large $x$ behavior of (24), the order of limiting transitions, $x \to \infty$ and $\alpha \to 0^+$, is crucial. As long as $x$ is large but finite, $a_2$ is a continuous function of $\alpha$, since $1 - e^{-2\alpha x}$ tends to zero as $\alpha \to 0^+$. However, if the $x \to \infty$ limit is taken first, (24)
becomes
\[ a_2 = \frac{1}{2\rho_L} (-1 - 2\alpha + 4) \quad \text{for} \quad \alpha \in [0, 1], \] (27)
so that \( a_2 \) is no longer continuous at \( \alpha = 0^+ \). One can easily convince oneself that the discontinuity is a direct consequence of dropping the \( l = 0 \) eigenstate, since, in the \( x \to \infty \) limit, its energy gap with respect to the ground state becomes infinite as soon as \( \alpha \neq 0 \). In other words, the effect of the excited state when \( x \) becomes large is simply to smoothen the discontinuity, but not to alter the essence of the thermodynamics. Its presence is only felt when \( \alpha \sim 1/x \).

Note that (27) is just (25) with \( \alpha - 2 \) substituted for \( \alpha \). That is, one shifts \( \alpha \) in (25) by 2, from \( \alpha \in [-1, 0] \) to \( \alpha \in [1, 2] \) (which is always legal, because in the singular gauge one has periodicity in \( \alpha \) with period 2) and one continues it beyond the fermionic point \( \alpha = 1 \), where no peculiarity exists, down to \( \alpha \in [0, 1] \). This is justified except near the Bose point, in a vicinity whose width tends to zero with the magnetic field tending to infinity, as discussed above.

In terms of the 2-anyon spectrum (relative + center of mass), and leaving aside the excited state, the analytic part of the spectrum \( (18-21) \), i.e. the LLL spectrum, rewrites when \( \alpha \in [-1, 0] \) as
\[ E_2 = 2\omega_t + (l_1 + l_2 - \alpha)(\omega_t - \omega_c) \] (28)
and when \( \alpha \in [0, 1] \) as
\[ E_2 = 2\omega_t + (l_1 + l_2 + 2 - \alpha)(\omega_t - \omega_c) \] (29)
where \( 0 \leq l_1 \leq l_2 \) are bosonic quantum numbers. Note that computing \( a_2 \) directly from the spectrum (28,29) reproduces (25,27), and when \( \alpha \to 1 \), (29) rightly becomes the 2-fermion spectrum in the LLL
\[ E_2 = 2\omega_t + (l'_1 + l'_2)(\omega_t - \omega_c) \] (30)
with \( 0 \leq l'_1 < l'_2 \).

What we have just learned from the solvable 2-anyon system can be expected to be valid for the \( N \)-anyon system as well. Namely, starting from the exact \( N \)-anyon spectrum \( (8,10) \) when \( \alpha \in [-1, 0] \), one can check that, by analogy with (28,29), when \( \alpha \in [0, 1] \),
\[ E_N = N\omega_t + \left[ \sum_{i=1}^{N} l_i + \frac{N(N-1)}{2}(2 - \alpha) \right](\omega_t - \omega_c) \] (31)
and
\[ \psi_{\text{free}} = \prod_{i<j} (z_i - z_j)^{2-\alpha} \prod_{i=1}^{N} z_i^{l_i} \exp\left(-\frac{1}{2} \omega_t \sum_{i=1}^{N} z_i \bar{z}_i \right) \] (32)
with \( 0 \leq l_1 \leq \ldots \leq l_N \), are eigenvalues and eigenstates of (4) taking into account the redefinition (3). When \( \alpha \to 1 \), both (31,32) describe \( N \) fermions with energy

\[
E_N = N\omega_t + (\omega_t - \omega_c) \left[ \sum_{i=1}^{N} \ell_i' \right]
\]

and \( 0 \leq \ell_1' < \ldots < \ell_N' \).

We claim that (31,32) captures the physics in the anti-screening regime, too, up to the effect of the unknown eigenstates whose role, in the large \( x \) limit, is only to smoothen the discontinuity in the equation of state when \( \alpha \to 0^+ \). The equation of state stemming from (31) is

\[
\beta P = \rho_L \ln(1 + \frac{\nu}{1 + (\alpha - 2)\nu})
\]

with a critical filling \( \nu_{\text{cr}} = 1/(2 - \alpha) \) where the pressure diverges, describing a nondegenerate ground state with all the \( \ell_i' \)'s null: in the singular gauge, \( \psi_{\text{free}} = \prod_{i < j} (z_i - z_j)^2 \exp(-\frac{\omega_c}{2} \sum_{i}^{N} z_i \bar{z}_i) \) (35) again becomes, when \( \alpha = 1 \), the fermionic Vandermonde determinant built from 1-body Landau eigenstates.

The physical situation in Fig. 3 is rather striking: Moving away from Bose statistics by attaching infinitesimal flux tubes anti-parallel to the magnetic field—the screening regime—leads to a smooth interpolation between Bose (\( \nu_{\text{cr}} = \infty \)) and Fermi (\( \nu_{\text{cr}} = 1 \)) statistics, whereas attaching infinitesimal statistical flux tubes parallel to the magnetic field—the anti-screening regime—condenses the system into a \( \nu_{\text{cr}} = 1/2 \) quantum system.

The equation of state taking the same form both in the screening and antiscreening regime, so should do its interpretation in terms of exclusion statistics. It is possible to adduce a simple semiclassical picture, within the approach of Ref. [15], that helps understand the metamorphosis with the exclusion statistics parameter and, by consequence, the filling factor. One starts with semiclassical single-particle orbits in the harmonic-Landau potential, which are formed by two normal modes with frequencies \( \omega_{\pm} = \pm \omega_t - \omega_c \). The “splitting” (\( \omega_+ \)) mode corresponds to the lifted degeneracy of the Landau levels, in particular of the LLL. Note that \( \omega_+ \to 0 \) when \( \omega \to 0 \). The Landau (\( \omega_- \)) mode corresponds to different Landau levels; thus, within the LLL it cannot be excited. The corresponding orbits are circles with opposite directions of rotation for the two modes, as evidenced by the opposite signs of \( \omega_+ \) and \( \omega_- \). Single-particle LLL orbits, without regard for statistical interaction, are concentric circles, with the value \( l \) of the angular momentum, an integer, corresponding to the number of quanta of the excited splitting mode. Bosons can all be in the ground state \( l = 0 \), fermions must occupy the \( l = 0, 1, \ldots \) states, one particle per
Figure 3: The critical filling as a function of $\alpha$. There is a discontinuity at $\alpha = 0^+$: the unknown nonlinear eigenstates which join the groundstate at $\alpha = 0^+$ smoothen the discontinuity.
state. Now consider two anyons in this picture. The first one can sit in the ground state, and, the second one being in an excited state with angular momentum $l$, their statistical exchange phase will be $\exp(i\pi l)$. Demanding this to be equal to $\exp(-i\pi\alpha)$ yields $l = -\alpha, 2 - \alpha, 4 - \alpha, \ldots$. For $\alpha$ negative, the lowest allowed angular momentum of the second particle is thus $l = -\alpha$, which corresponds to exclusion statistics with statistics parameter $g = -\alpha$ (the presence of the first particle excludes $g$ states, in terms of the values of the quantum number, for the second one). However, for $\alpha$ positive, the would-be lowest value of $l$ is negative, which is prohibited: it corresponds to the opposite direction of rotation, that is, to an excitation of the Landau mode. This is, in a different language, precisely the same thing as the two-particle ground state detaching from the LLL basis [cf. (22)]. The true lowest allowed value of $l$ belonging to the splitting mode for $\alpha > 0$ is therefore $2 - \alpha$, and the same is the exclusion statistics parameter: $g = 2 - \alpha$. This corresponds exactly to the equation of state (34). Imposing the condition that the Landau mode may not be excited at all corresponds to taking the $\omega_c \to \infty$ limit first; the whole construction is then valid for any $\alpha \in [0, 1]$.

To deduce the complete spectrum, one still has, in fact, to refer to the underlying quantum-mechanical problem. In particular, the semiclassical picture alone will not explain why $l = 3 - \alpha$ is allowed for the second particle (it has to be in order to get the count of excited states right) while $l = 1 - \alpha$ is not; the answer is that quantum-mechanically, those contain the $l_{\text{CM}} = 1$ center-of-mass excitations over, respectively, $l_{\text{rel}} = 2 - \alpha$ (which is allowed) and $l_{\text{rel}} = -\alpha$ (which is not). The inherent problem with the semiclassical picture per se is that single-particle angular momenta are not good quantum numbers for anyons. However, with the quantum-mechanical knowledge put in as outlined, a generalization to $N$ particles is possible and yields the exact result in the LLL anti-screening regime, like it does in the screening regime [15]. Semiclassically, the third particle has the first ($l = 0$) and second ($l = 2 - \alpha$) ones inside its orbit and therefore has to have its angular momentum equal to $n - 2\alpha$, to provide for the correct Aharonov-Bohm phase. Now, $n = 3$ is excluded because quantum-mechanically, that would involve as one of the states in the superposition the state with the relative angular momentum of the second and third particles equal to $-\alpha$, which is not allowed (Landau excitation), leaving $4 - 2\alpha$ as the lowest possible angular momentum for the third particle. Continuing to $N$ particles, the ground state energy $E_{0N} = \frac{1}{2}N(N-1)(2-\alpha)(\omega_t - \omega_c)$ is thus correctly reproduced, up to the constant shift. The crucial difference in the critical filling factor between the two directions of the infinitesimal flux is thereby interpreted in terms of exclusion statistics: With screening, adding a particle excludes only $\alpha$ quantum states because the sign of the angular momentum remains the one that belongs to the LLL, but with anti-screening, the sign gets reversed and the would-be ground state is promoted to the next Landau
level, which is why the whole of $2 - \alpha$ (the next allowed value of the relative angular momentum) states get excluded.

By way of supporting this claim quantitatively, it makes sense, as a first step, to look at the 3-anyon problem and its thermodynamics, described by the third virial coefficient $a_3$, or by the third cluster coefficient $b_3$ from which $a_3$ is deduced. One has linear eigenstates (13,14), which generalize the linear eigenstates (15,17) of the 2-body problem, and nonlinear eigenstates, which are only known numerically (10). In the absence of a magnetic field, the latter have been shown (16) to render $\omega_{\text{lin}} = 0$, where 1 denotes a ground state.

Restricting oneself to the contribution of the linear states, one obtains for $\alpha \in [0, 2]$

$$b_3^{\text{lin}} = \frac{e^{-3(\tilde{x} + x)}}{(1 - e^{-\tilde{x}})(1 - e^{-\tilde{x} - 2x})} \left[ \frac{e^{-3\tilde{x}(2 - \alpha)}}{(1 - e^{-2\tilde{x}})(1 - e^{-3\tilde{x}})(1 - e^{-2\tilde{x} - 2x})(1 - e^{-3\tilde{x} - 2x})} \right. $$

$$- \frac{1}{(1 - e^{-2\tilde{x}})(1 - e^{-\tilde{x}})(1 - e^{-2\tilde{x} - 2x})(1 - e^{-\tilde{x} - 2x})} + \frac{1}{3(1 - e^{-\tilde{x}})^2(1 - e^{-2\tilde{x} - 2x})^2} \left. \right]$$

$$+ \frac{1}{(1 - e^{2\tilde{x}})(1 - e^{2\tilde{x} + 2x})(1 - e^{2\tilde{x} + 4x})(1 - e^{3\tilde{x} + 6x})(1 - e^{2\tilde{x} + 2x})(1 - e^{3\tilde{x} + 4x})}$$

$$- \frac{1}{(1 - e^{\tilde{x}})(1 - e^{\tilde{x} + 2x})(1 - e^{2\tilde{x} + 4x})(1 - e^{2\tilde{x} + 2x})(1 - e^{-\tilde{x}})(1 - e^{-3\tilde{x} - 2x})}$$

where $\tilde{x} = \beta(\omega_l - \omega_c)$. Taking in (33) the thermodynamic limit, whence $\tilde{x} \rightarrow (\beta \omega)^2/2x$, one sees that $b_3^{\text{lin}}$ exhibits an unphysical volume divergence at leading order $1/\tilde{x}^2$ which is obviously due to the dropping of the infinite set of unknown nonlinear eigenstates. In the LLL limit $x \rightarrow \infty$, (33) becomes

$$b_3^{\text{lin}} \simeq e^{-3x} \left[ -\frac{1}{\tilde{x}^2} e^{-2\alpha x} + \frac{1}{18x} (9(2 - \alpha)^2 - 9(2 - \alpha)(1 + 2e^{-2\alpha x}) + 2(1 + 9e^{-6\alpha x} + 36e^{-2\alpha x}) \right]$$

(37)

Again, the unphysical $1/\tilde{x}^2$ volume divergence would not appear if the nonlinear states were included. We are interested in the term with the $1/\tilde{x}$ volume divergence which makes, in the thermodynamic limit, the cluster coefficient proportional to the volume, as it should be. When this term is periodically extended from $\alpha \in [1, 2]$ onto $\alpha \in [-1, 0]$, a discontinuity similar to the one described in the 2-anyon case arises. Nicely enough, both the unphysical volume divergence and the LLL discontinuity are controlled by the

\[10\]
very same exponential factor $e^{-2\alpha x}$ which was already operative in the 2-anyon problem. Putting all these considerations together, we obtain, in the thermodynamic limit [17] $1/\tilde{x} \to 3\rho_L V$ and in the large magnetic field limit, for $\alpha \in [-1, 0]$

$$b_3 = \rho_l V e^{-3x} \frac{(3\alpha + 1)(3\alpha + 2)}{3!}$$

(38)

which is consistent with (12), and for $\alpha \in [0, 1]$

$$b_3 = \rho_l V e^{-3x} \frac{(3(\alpha - 2) + 1)(3(\alpha - 2) + 2)}{3!}$$

(39)

which is consistent with (31) and (34). Again, the effect of the linear and nonlinear states joining the LLL at $\alpha = 0^+$ would amount to smoothening this discontinuity at the Bose point. Clearly, a generalization of these results to the $N$-anyon case should follow the same lines of reasoning.

To conclude, let us recapture our main claim again. The LLL anyon equation of state is continuous through the Fermi point but behaves in two much different ways near the Bose point, depending on which side the latter is approached from. In the screening regime, it tends to the bosonic equation in a smooth manner, with the critical filling factor going to infinity as $\alpha \to 0^-$. In the anti-screening regime, however, there is an abrupt change on a narrow interval near the Bose point due to extra states joining the LLL at $\alpha = 0^+$. If the $B \to \infty$ limit is taken first, that is, if one ignores these extra states, an unphysical discontinuity arises, and the critical filling factor at $\alpha \to 0^+$ then tends to 1/2. In reality, at no matter how small positive $\alpha$, the critical filling tends to $1/(2 - \alpha)$ as $B \to \infty$. It being believed that the $\nu_{cr} = 1/2$ state should play an important role in the fractional quantum Hall effect (see for example the composite fermion approach and the resulting Jain series [18]), the LLL-anyon model provides a scenario of how the 1/2 filling may arise without relying on extra interaction (like the Coulomb interaction that plays a crucial role in the usual picture of FQH states), but just from the interplay of the strong magnetic field and statistics close to the Bose point in the anti-screening regime.

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References

[18] see, for example, J.K. Jain and R.K. Kamilla, cond-mat/9704031.