ANYONIC PARTITION FUNCTIONS
and
WINDINGS OF PLANAR BROWNIAN MOTION

Jean DESBOIS, Christine HEINEMANN and Stéphane OUVRY

Division de Physique Théorique, IPN, Orsay Fr-91406

Abstract: The computation of the $N$-cycle brownian paths contribution $F_N(\alpha)$ to the $N$-anyon partition function is addressed. A detailed numerical analysis based on random walk on a lattice indicates that $F_N^{(0)}(\alpha) = \prod_{k=1}^{N-1}(1 - \frac{N}{k} \alpha)$. In the paramount 3-anyon case, one can show that $F_3(\alpha)$ is built by linear states belonging to the bosonic, fermionic, and mixed representations of $S_3$.

IPNO/TH 94-55
June 1994

\textsuperscript{1} and LPTPE, Tour 12, 3ème étage, Université Paris 6, 75005 Paris Cedex / electronic e-mail: OUVRY@FRCPN11
\textsuperscript{2} Unité de Recherche des Universités Paris 11 et Paris 6 associée au CNRS

1
In this note one considers 2-dimensional identical particles with fractional statistics (anyons) [1]. This system seems to play an important role in the understanding of the Fractional Quantum Hall Effect [2].

The $N$-anyon Hamiltonian (there is no additional mutual interaction) reads

$$H_N = \frac{1}{2m} \sum_{i=1}^{N} (\vec{p}_i - \alpha \vec{A}_i)^2 + \sum_{i=1}^{N} \frac{1}{2} m \omega^2 r_i^2$$

where $\alpha$ is the statistical parameter ($\alpha = 0$ corresponds to the bosonic case, $\alpha = 1$ to the fermionic case), and $\vec{A}_i$ the statistical vector potential. An harmonic potential term has been added as a long distance regulator. It discretizes the energy spectrum, and allows for a convenient computation of thermodynamical quantities in the thermodynamic limit ($\omega \to 0$, where $\omega$ is the harmonic oscillator pulsation).

In a recent series of papers [3], the $N$-anyon partition function has been shown to be written as

$$Z_N(\alpha) = \sum_\mathcal{P} F_\mathcal{P}(\alpha) \prod_L \frac{1}{\nu_L!} \left( \frac{Z_1(L\beta)}{L} \right)^{\nu_L}$$

where summation is understood over all partitions $\mathcal{P}$ of $N$. Each partition $\sum_L L \nu_L = N$ is in one-to-one correspondence with an equivalence class of the permutation group $S_N$, namely the class of permutations which can be written as a product of cycles of length $L$, $\nu_L$ times. The one-particle partition function $Z_1(\beta) = 1/(2 \sinh \frac{\xi}{2})^2$ is obviously statistics independant ($\xi = \hbar \beta \omega, \beta = \frac{1}{kT}$). Note that in (2), the statistical information has been completely factorized out in the $F_\mathcal{P}(\alpha)$’s, which satisfy

$$F_\mathcal{P}(1 + \alpha) = sgn(\mathcal{P}) F_\mathcal{P}(\alpha)$$

It follows that $Z_N(\alpha)$ is naturally written as a sum of symmetric and antisymmetric terms under $\alpha \to 1 + \alpha$. By convention one has $F_\mathcal{P}(0) = 1$ for all $\mathcal{P}$, i.e. $\alpha = 0$ corresponds to Bose statistics.
In a brownian motion language, the \( F_p \)'s are simply expressed as

\[
F_p(\alpha) = \langle e^{i\alpha \sum_{i<j} \theta_{ij}} \rangle_{(C)}
\]  

(4)

where the average \( \langle \rangle_{(C)} \) is taken over all possible brownian paths which induce, in the configuration space, a given permutation \( P \), i.e. which correspond to a given partition of the integer \( N \). Note that these curves are closed curves in the configuration space quotiented by the permutation group \( S_N \). The \( \theta_{IJ} \)'s are the relative angles between any pair of particles \( \{I,J\} \) (see below for a more precise definition). For example, \( F_N \) is associated with cycles of maximal length \( L = N \), with \( \nu_L = 1 \). Such cycles, involving the \( N \) particles of the system, are by definition \( N \)-particles paths over a time interval \( \tau = h\beta \). Equivalently, they can be considered as 1-particle paths over a time interval \( N\tau \), simply because the particles are identical \([3]\). Therefore, \( \langle \rangle_{(C)} \) can be taken over all brownian paths of length \( N\tau \). These paths are divided into \( N \) segments of equal length \( \tau \), each segment corresponding to the motion of a point \( M_I \) (all the \( M_I \)'s have the same speed).

The \( \theta_{IJ} \)'s are defined as the angles of the vectors \( \vec{M}_I \vec{M}_J \) with a fixed axis in the plane \( (-\infty < \theta_{IJ} < +\infty) \) (see fig. 1 for \( F_3 \)).

In the simplest \( N = 2 \) case, eq. (2), has been exactly solved \([1]\). It is instructive to display the results and their intimate connection with probability distributions of planar brownian windings. \( Z_2(\alpha) \) is given as

\[
Z_2(\alpha) = \frac{1}{2} F_2(\alpha) Z_1(2\beta) + \frac{1}{2} F_{11}(\alpha) Z_1(\beta)^2
\]  

(5)

and is computed from the complete linear states

\[
\psi_{n,m}(r, \theta) = e^{in\theta} e^{-\beta r^2} e^{im-\alpha} |L_n^{m-\alpha}|(\beta r^2)
\]  

(6)

\[
E_{n,m} = \omega(2n + |m-\alpha| + 1)
\]  

(7)
where the $L_n$’s are the Laguerre polynomials. One gets for $F_2(\alpha)$ and $F_{11}(\alpha)$

$$F_2(\alpha) = \frac{\sinh(1 - 2\alpha)\xi}{\sinh \xi} \quad F_{11}(\alpha) = \frac{\cosh(1 - 2\alpha)\xi}{\cosh \xi}$$  \hspace{1cm} (8)

In terms of brownian winding, the Fourier transform of $F_{11}(\alpha)$ \((0 < \alpha < 1)\) yields the probability for a brownian curve to wind an even number of half-windings around a fixed point, i.e. the usual result for an integer brownian winding \([4]\). $F_2(\alpha)$ leads to the probability for an odd number of half-windings \([3]\).

The second virial coefficient follows directly \([5]\)

$$a_2(\alpha) = \frac{1}{4} - \frac{(1 - \alpha)^2}{2}$$  \hspace{1cm} (9)

One notes that in the thermodynamic limit where the harmonic potential vanishes

$$F_2(\alpha) \xrightarrow{\omega \rightarrow 0} F^{(0)}_2(\alpha) = 1 - 2\alpha$$  \hspace{1cm} (10)

$$F_{11}(\alpha) \xrightarrow{\omega \rightarrow 0} F^{(0)}_{11}(\alpha) = 1$$  \hspace{1cm} (11)

The trivial thermodynamical limit of $F_{11}(\alpha)$ is easily understandable in terms of \((4)\), since in this limit there is no chance for the 2 independent 1-particle brownian paths to overlap.

In the $N = 3$ case, things become non trivial, since only a part of the spectrum (the linear states \([6]\) that generalize \((6)\)) is known. One has by definition

$$Z_3(\alpha) = \frac{1}{3} F_3(\alpha) Z_1(3\beta) + \frac{1}{2} F_{21}(\alpha) Z_1(2\beta) Z_1(\beta) + \frac{1}{6} F_{111}(\alpha) Z_1(\beta)^3$$  \hspace{1cm} (12)

It happens that the antisymmetric part of $Z_3(\alpha)$, namely $F_{21}(\alpha)$, is known \([7]\) to be exactly built from the linear states, i.e.

$$Z_1(\beta) Z_1(2\beta) F_{21}(\alpha) = Z_3^{lin}(\alpha) - Z_3^{lin}(\alpha + 1)$$  \hspace{1cm} (13)
This yields
\[ F_{21}(\alpha) = \frac{\sinh(1 - 2\alpha) \xi}{\sinh \frac{3\xi}{2}} \] (15)

with
\[ F_{21}(\alpha) \xrightarrow{\omega \to 0} F_{2}^{(0)}(\alpha) \] (16)
as it should.

In [3], numerical estimations for \( F_{3}^{(0)}(\alpha) \), and for \( F_{111}(\alpha) \) at order \( \xi^4 \), have been performed in the thermodynamic limit
\[ F_{3}^{(0)}(\alpha) \sim (1 - 3\alpha)(1 - \frac{3\alpha}{2}) \] (17)

This numerical result, altogether with \( F_{111}^{(4)}(\alpha) \), is sufficient to propose a simple analytical expression for the third virial coefficient [3]
\[ a_3(\alpha) \sim \frac{1}{36} + \frac{1}{12\pi^2} \sin^2(\pi\alpha) \] (18)

Having reached this point, and looking at the simple expression for \( F_{3}^{(0)}(\alpha) \), it seems natural to suspect that it could be generalized to \( N > 3 \), and that it might be derived from pure theoretic considerations. This is what is now going to be shown.

Indeed, using (4), one is tempted to make numeric simulations on a lattice for the \( F_{P}(\alpha) \)'s. Note that in the thermodynamic limit, one has always that the \( F_{P}^{(0)}(\alpha) \)'s factor out in a product of \( F_{L}^{(0)}(\alpha) \)'s, where the \( L \)'s enter in the partition \( P \). From this consideration, it follows that only the \( F_{N}(\alpha) \)'s should be easily accessible numerically. Note also that the regularisation due to the lattice spacing necessarily alters the results, since two points cannot approach each other arbitrarily close ([8,9]). In the case of a true brownian
motion, when two points \( M_I \) and \( M_J \) come close to each other, the accumulated angle \( \theta_{IJ} \) has a non negligible probability to become arbitrarily large (the typical brownian path being non-differentiable, \( \theta_{IJ} \) follows a broad law probability distribution). Thus, when one tries to simulate brownian paths on a lattice, a correction is necessary to take into account this characteristics of brownian winding. Each time two points \( M_I \) and \( M_J \) randomly walking on the lattice are separated by only one lattice spacing, one adds to \( \theta_{IJ} \) a multiple of \( 2\pi \) which is drawn accordingly to a Cauchy law \([9]\). Typically, we have used random walks ranging from 50000 to 150000 steps. The resulting numeric simulations for the \( F_N^{(0)}(\alpha) \)'s suggest the following simple formula

\[
F_N^{(0)}(\alpha) = \prod_{k=1}^{N-1} \left( 1 - \frac{N}{k^\alpha} \right) \quad (19)
\]

valid for \( 0 < \alpha < 1 \) (to obtain an expression in the whole interval \([0, 2]\) one uses \( F_N^{(0)}(\alpha) = F_N^{(0)}(-\alpha) = F_N^{(0)}(\alpha + 2) \)).

Eq. (19) correctly reproduces the \( N = 2 \) and \( N = 3 \) cases ; it also has the right \( \alpha = 0 \) and \( \alpha = 1 \) limits ; finally it satisfies \( F_N(1 + \alpha) = (-1)^{N+1} F_N(\alpha) \). Note that a similar formula has recently appeared in an a priori different context \([10]\), the one-dimensional Calogero model \([11]\). There, the \( F_N^{(0)}(\alpha) \)'s, are derived from the exact one dimensionnal \( N \)-body spectrum (they are the coefficients appearing in the expansion of the density in powers of the fugacity). This similarity is not a surprise : it is well-known that there is a correspondence \([12]\) between the Calogero model and the anyon model projected into the lowest Landau level of an external magnetic field. This might imply that when anyons are projected into the Landau groundstate, mostly cycles of maximal length contribute to the thermodynamics.

The expression obtained for \( F_N^{(0)}(\alpha) \) is again quite simple. One should be able to find a way to support it by a pure analytical approach. Let us concentrate on the \( N = 3 \) case.
As already stated, the antisymmetric part $F_{21}(\alpha)$ of $Z_3(\alpha)$ is determined by the known linear states; its symmetric part is

$$Z_3(\alpha) + Z_3(1 + \alpha) = \frac{2}{3} F_3(\alpha) Z_1(3\beta) + \frac{1}{3} F_{111}(\alpha) Z_1(\beta)^3$$

(20)

Could it be that $F_3$ is also determined by the known part of the 3-anyon spectrum? Some simple considerations [13] on $F_{111}$ will show us the way. The paths contributing to $F_{111}(\alpha)$ are simple one-particle cycles going back to their initial position at the time $\tau = \hbar \beta$. The final configuration is identical to the initial one, thus there is no need for quotienting the configuration space by the permutation group $S_3$. It follows that $Z_1(\beta)^3 : F_{111}(\alpha)$ is simply the partition function of a system of non-identical particles (Boltzmann statistics) interacting topologically à la Aharonov-Bohm with coupling constant $\alpha$ now defined modulo 1 instead of modulo 2 in the identical particle case. In the case of 2-anyon one indeed has that

$$Z_1(\beta)^2 : F_{11}(\alpha) = Z_2(\alpha) + Z_2(\alpha + 1)$$

(21)

meaning that $F_{11}(\alpha)$ (for 2 non-identical particles) can be directly obtained from $Z_2(\alpha)$ (the partition function for 2 bosonic identical particles)[3]. Thus one writes for $F_{111}$

$$Z_1(\beta)^3 : F_{111}(\alpha) = Z_3(\alpha, \alpha, \alpha) + Z_3(\alpha + 1, \alpha + 1, \alpha + 1) + 2Z_3(\text{mixed})$$

(22)

with

$$Z_3(\text{mixed}) = Z_3(\alpha, \alpha, \alpha + 1) + Z_3(\alpha, \alpha + 1, \alpha + 1)$$

(23)

where one has explicitly denoted the relative statistical coefficient for each pair of particles (here $\alpha$ or $\alpha + 1$). In the case $\alpha = 0$, this implies that the mixed representation of $S_3$ is

---

[3] In other words in $Z_2(\alpha)$ one has summed on even angular momentum quantum numbers, and, in order to get the complete spectrum, one simply shifts $\alpha \rightarrow \alpha + 1$ to get the odd angular momentum quantum numbers.
represented either by (0, 0, 1) or by (0, 1, 1), meaning in the first case that the pairs \(\{1, 2\}\) and \(\{1, 3\}\) are bosonic, and \(\{2, 3\}\) is fermionic, and in the second case that the pair \(\{1, 2\}\) is bosonic, whereas \(\{1, 3\}\) and \(\{2, 3\}\) are fermionic. The factor 2 in front of \(Z_3(mixed)\) is due to the fact that the symmetric and antisymmetric representations of \(S_3\) are one-dimensional, but the mixed one is two-dimensional implying doubly degenerated states in the spectrum.

This in turn implies for \(F_3(\alpha)\)

\[
Z_1(3\beta)F_3(\alpha) = Z_3(\alpha, \alpha, \alpha) + Z_3(\alpha + 1, \alpha + 1, \alpha + 1) - Z_3(mixed)
\] (24)

This formula is analogous to (13), so one might ask about its validity when one restricts oneself to linear states. One has the generalized linear eigenstates [14]

\[
\psi_{n,m_{IJ}}(r_{IJ}, \theta_{IJ}) = e^{-\beta \rho^2} \prod_{I < J} e^{im_{IJ}\theta_{IJ} - \alpha_{IJ}} L_n^{N-2+\sum_{I < J}^{|m_{IJ} - \alpha_{IJ}|}} (\beta \rho^2) \] (25)

\[
E_{n,m_{IJ}} = \omega(2n + \sum_{I < J}^{|m_{IJ} - \alpha_{IJ}|} + N - 1) \] (26)

for a topological Hamiltonian where the Aharonov-Bohm coupling constants \(\alpha_{IJ}\) depend on the pair \(\{I, J\}\) of particles coupled \((\rho^2 = \sum_{I < J} r_{IJ}^2)\). Mixed linear states are obtained by trading for each couple \(\{I, J\}\) of particles the statistical parameter \(\alpha\) for a statistical parameter \(\alpha_{IJ}\). One can easily show that the partition function for such mixed linear states is then directly obtained from the linear partition function \(Z_3^{lin}(\alpha)\) by \(\frac{N(N-1)}{2} \alpha : \sum_{I < J} \alpha_{IJ}\). For \(N = 3\)

\[
Z_3^{lin}(\alpha, \alpha, 1 + \alpha) = \frac{\cosh[(2-3\alpha)\xi]}{32 \sinh^2 \frac{3\xi}{2} \sinh \xi \sinh^2 \frac{\xi}{2}}
\]

\[
Z_3^{lin}(\alpha, 1 + \alpha, 1 + \alpha) = \frac{\cosh[(1-3\alpha)\xi]}{32 \sinh^2 \frac{3\xi}{2} \sinh \xi \sinh^2 \frac{\xi}{2}}
\] (27)

Let us now consider (24) restricted to linear states only. One finds that if the expression

\[
Z_3^{lin}(\alpha, \alpha, \alpha) + Z_3^{lin}(\alpha + 1, \alpha + 1, \alpha + 1) - Z_3^{lin}(\alpha, \alpha, 1 + \alpha) - Z_3^{lin}(\alpha, 1 + \alpha, 1 + \alpha)
\] (28)
has not the correct finite limit $F_3^{(0)}(\alpha)$ in the thermodynamic limit, but on the contrary diverges, the left over divergence does not depend on $\alpha$ and simply reads $Z_1(\beta)Z_3(3\beta)$. It is quite a striking fact that when one adds $Z_1(\beta)Z_3(3\beta)$ to $Z_3^{lin}(\alpha, \alpha, \alpha + 1) + Z_3^{lin}(\alpha, \alpha + 1, \alpha + 1)$ in (28) one gets in the thermodynamic limit exactly the right $F_3^{(0)}(\alpha)$ as given in (17). Moreover, one can compute $F_3(\alpha)$ as given by (28) for an arbitrary $\xi$

$$F_3(\alpha) = \frac{\sinh(2 - 3\alpha)\xi}{\sinh 3\xi} \frac{\sinh(1 - 3\alpha)\xi}{\sinh 3\xi}$$

(29)

This expression generalizes quite nicely $F_2(\alpha)$ given in (8).

Similar considerations (albeit less transparent) can be made for the $F_N^{(0)}$s, $N > 3$. One notes that a possible generalization of (29)

$$F_N(\alpha) = \prod_{k=1}^{N-1} \left( \frac{\sinh(k - N\alpha)\xi}{\sinh 3\xi} \right)$$

(30)

would automatically satisfy the right thermodynamic limit (19) obtained in the numerical simulations.

To conclude this analysis, let us come back to the standard case $\alpha = 0$. One knows that $Z_1^3(\beta)F_{111}(0) = Z_3^3(\beta)$ simply describes 3 independent bidimensionnal oscillators; one also has

$$Z_3(0, 0, 0) = \frac{\cosh(3\xi) + 2\cosh^2(\xi/2)}{32\sinh^2 \frac{3\xi}{2} \sinh^2 \xi \sinh^2 \frac{3\xi}{2}}$$

(31)

$$Z_3(1, 1, 1) = \frac{1 + 2\cosh^2(\xi/2)}{32\sinh^2 \frac{3\xi}{2} \sinh^2 \xi \sinh^2 \frac{3\xi}{2}}$$

(32)

One would like to find simple expressions for $Z_3(0, 0, 1)$ and $Z_3(0, 1, 1)$. It is clear from the bosonic (31) and fermionic (32) 3-body partition functions that the linear part of the spectrum contributes respectively to $\cosh(3\xi)/d$ and $1/d$ (i.e. $\cosh(3 - 3\alpha)\xi/d$ for $\alpha = 0, 1$) whereas the common $2\cosh^2(\xi/2)/d$ is built by the unknown part of the spectrum.
\[ d = 32 \sinh^2 \frac{\xi}{2} \sinh^2 \xi \sinh^2 \frac{3\xi}{2} \]. It is quite natural to try for \( Z_3(0,0,1) \) and \( Z_3(0,1,1) \)

\[
Z_3(0,0,1) \sim \frac{\cosh(2\xi) + 2 \cosh^2(\xi/2)}{32 \sinh^2 \frac{\xi}{2} \sinh^2 \xi \sinh^2 \frac{3\xi}{2}} \tag{33}
\]

\[
Z_3(0,1,1) \sim \frac{\cosh \xi + 2 \cosh^2(\xi/2)}{32 \sinh^2 \frac{\xi}{2} \sinh^2 \xi \sinh^2 \frac{3\xi}{2}} \tag{34}
\]

and to check if (22) holds for \( \alpha = 0 \). One finds again the same left over divergence \( Z_1(\beta)Z_1(3\beta) \) as above (eventhough one gets the correct behavior \( \sim \frac{1}{\xi^4} \) in the thermodynamic limit), that should be added to \( Z_3(0,0,1) + Z_3(0,1,1) \) to get \( Z_1(\beta)^3 \). It would certainly be interesting to understand more precisely, already at the level of (22), the origin of this additional term, and also in which way it splits between \( Z_3(0,0,1) \) and \( Z_3(0,1,1) \), thus allowing a complete knowledge of the 3-body partition functions for the mixed eigenstates of \( S_3 \).

Acknowledgements : J. D. and S. O. acknowledge stimulating conversations with J. Myrheim, and also for drawing our attention on [10].

Figure Captions :

A brownian closed path contributing to \( F_3^{(0)}(\alpha) \). The points A,B and C divide the curve into 3 patches of equal length. The 3 points \( M_1, M_2 \) and \( M_3 \) span each of the patches with an equal speed (\( M_1 \) goes from A to B, etc...).
References: