DIFFUSIVE TRANSPORT IN A ONE DIMENSIONAL DISORDERED POTENTIAL INVOLVING CORRELATIONS

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Abstract

This article deals with transport properties of one dimensional Brownian diffusion under the influence of a correlated quenched random force, distributed as a two-level Poisson process. We find in particular that large time scaling laws of the position of the Brownian particle are analogous to the uncorrelated case. We discuss also the probability distribution of the stationary flux going through a sample between two prescribed concentrations, which differs significantly from the uncorrelated case.

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Disorder significantly influences diffusive transport phenomena [1] [2] [3]. Indeed, strong enough disorder does not induce a simple renormalization for transport coefficients of the corresponding pure system, but usually generates "anomalous diffusion" effects. The case of spatially uncorrelated disordered media is now pretty well understood. The effects of spatial correlations on the dynamics have first been discussed for the directed random walk problem [4]. More recently, the response to an external applied force has been studied for the one dimensional case of a Brownian particle diffusing under the influence of a quenched random force $\{F(x)\}$ [5]. For a sample characterized by some particular realization of the stochastic process $\{F(x)\}$, the diffusion is defined by the following Fokker-Planck equation for the probability density $P(x,t\,|x_0,0)$

$$\frac{\partial P}{\partial t} = D_0 \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} - \beta F(x) P\right) \tag{1}$$

In this article, we analyse in details transport properties of this model in the case where $\{F(x)\}$ is distributed as a two-level Poisson process.

Let us define more precisely the model of disorder we consider and some of its properties. We assume that the quenched random force $F(x)$ takes alternatively a positive value $\phi_0 > 0$ and a negative value $-\phi_1 < 0$ on intervals whose lengths are independent random variables distributed according to the following probability densities respectively

$$\begin{cases} f_0(l) = \theta(l)\ n_0\ e^{-n_0 l} \\ f_1(h) = \theta(h)\ n_1\ e^{-n_1 h} \end{cases} \tag{2}$$

The parameters $\frac{1}{n_0}$ and $\frac{1}{n_1}$ are respectively the mean length of intervals $\{F(x) = \phi_0\}$ and the mean length of intervals $\{F(x) = -\phi_1\}$. This choice of exponential distributions for f_0 and f_1 is in fact the only one that makes the process $\{F(x)\}$ Markovian. This property enables us to write differential equations for the probability $p_0(x)$ to have $F(x) = \phi_0$ and the probability $p_1(x) = 1 - p_0(x)$ to have $F(x) = -\phi_1$.

$$\begin{cases} \dfrac{\partial p_0}{\partial x} = -n_0 p_0 + n_1 p_1 = n_1 - (n_0 + n_1) p_0 \\ \dfrac{\partial p_1}{\partial x} = -n_1 p_1 + n_0 p_0 = n_0 - (n_0 + n_1) p_1 \end{cases} \tag{3}$$

The mean value F_0 and the two point correlation function $G(x)$ of the process $\{F(x)\}$ read

$$\begin{cases} F_0 \equiv <\phi> = \phi_0 \dfrac{n_1}{n_0 + n_1} - \phi_1 \dfrac{n_0}{n_0 + n_1} \\ G(x) \equiv <\phi(x)\phi(0)> - <\phi>^2 = \dfrac{n_0 n_1}{(n_0 + n_1)^2} (\phi_0 + \phi_1)^2 e^{-(n_0+n_1)|x|} \end{cases} \tag{4}$$

The corresponding random potential $U(x) = -\int^x F(y)dy$ seen by the Brownian particle presents an alternance of positive and negative slopes of random lengths as sketched on Figure 1.

The fundamental random variable associated with classical diffusion under the action of a quenched random force presenting a stricly positive mean $< F(x) >\equiv F_0 > 0$, is the exponential functional of the random potential $U(x)$

$$\tau_\infty = \int_0^\infty dx\ e^{\beta U(x)} = \int_0^\infty dx\ e^{-\beta \int_0^x F(y)dy} \tag{5}$$

Indeed, the probability distribution of the functional τ_∞ determines the large time anomalous behaviour of the Brownian particle position [3]. In particular, the velocity defined for each sample as

$$V = \lim_{t\to\infty} \frac{d}{dt} \int_{-\infty}^{+\infty} dx\ xP(x,t\ |x_0,0) \tag{6}$$

is a self-averaging quantity [6] (for any homogenous random potential presenting only short-range correlations) inversely proportional to the first moment of τ_∞ [3]

$$V = \frac{D_0}{< \tau_\infty >} \tag{7}$$

When the quenched random force is distributed with the Gaussian measure

$$\mathcal{D}F(x)\ e^{-\frac{1}{2\sigma}\int dx\ [F(x)-F_0]^2} \tag{8}$$

the probability distribution $\mathcal{P}_\infty(\tau)$ of the functional τ_∞ reads in terms of $\alpha = \frac{\sigma\beta^2}{2}$ and $\mu = \frac{2F_0}{\beta\sigma} > 0$ [3]

$$\mathcal{P}_\infty(\tau) = \frac{\alpha}{\Gamma(\mu)} \left(\frac{1}{\alpha\tau}\right)^{1+\mu} e^{-\frac{1}{\alpha\tau}} \underset{\tau\to\infty}{\propto} \frac{1}{\tau^{1+\mu}} \tag{9}$$

The algebraic decay at large τ explains all the dynamical phases transitions between different anomalous behaviours known for this model [3]. In particular, Eq (7) implies that the value $\mu = 1$ separates a phase of vanishing velocity $V = 0$ for $0 < \mu < 1$, and a phase of finite velocity $V > 0$ for $\mu > 1$. Another interesting physical quantity is the stationnary current J_N which goes through a disordered sample of length N between a fixed concentration P_0 and a trap described by the boundary condition $P_N = 0$ [7] [8]. In the limit $N \to \infty$, the stationary flux J_∞ is simply a random variable inversely proportional to the functional τ_∞

$$J_\infty = \frac{D_0 P_0}{\tau_\infty} \tag{10}$$

Note that unlike the velocity V, this flux is not a self-averaging quantity, but must be described by its full probability distribution.

It is very convenient to introduce the more general functional

$$\tau(x,b) = \int_x^b dy\ e^{\beta\,[U(y)-U(x)]} = \int_x^b dy\ e^{-\beta\int_x^y F(u)du} \tag{11}$$

and to consider the random variable τ_∞ as the limit of this process as $x \to -\infty$

$$\tau_\infty = \lim_{x\to-\infty}\ \tau(x,b) \tag{12}$$

The evolution of the functional $\tau(x,b)$ is governed by the the stochastic differential equation

$$\frac{\partial\tau}{\partial x} = \beta F(x)\tau(x,b) - 1 \tag{13}$$

The stochastic term $F(x)$ appears multiplicatively, so that the fluctuations of the random force are coupled to the values taken by the random process $\tau(x,b)$ itself. When the random force $\{F\}$ is distributed as a white noise (8), the multiplicative stochastic process $\tau(x,b)$ can be in fact be related to Brownian motion on a surface of constant negative curvature [9].

We now compute the probability distribution of the functional τ_∞ when the quenched random force $\{F(x)\}$ is a two-level Poisson process, and compare it with the result (9) for the white noise case. The random variable τ_∞ only exists if the force mean value F_0 is strictly positive

$$(n_0+n_1)F_0 = \phi_0 n_1 - \phi_1 n_0 > 0 \qquad \text{that is} \quad \frac{n_1}{\phi_1} - \frac{n_0}{\phi_0} > 0 \tag{14}$$

that we assume from now on. Note that the random variable $\tau(x,b)$ remains confined in an interval $[\tau_{min}(x,b), \tau_{max}(x,b)]$ depending on the length $(b-x)$

$$\left\{\begin{array}{l} \tau_{min}(x,b) = \int_x^b dy\ e^{-\beta\phi_0(y-x)} = \dfrac{1-e^{-\beta\phi_0(b-x)}}{\beta\phi_0} \\[2ex] \tau_{max}(x,b) = \int_x^b dy\ e^{+\beta\phi_1(y-x)} = \dfrac{e^{\beta\phi_1(b-x)}-1}{\beta\phi_1} \end{array}\right\} \tag{15}$$

This is very different from the white noise case, where the random force is not bounded. Even in the limit where the interval length $(b-x)$ tends to infinity, the support of the random variable τ_∞ is not $[0,+\infty]$, but $[\frac{1}{\beta\phi_0},+\infty]$.

The two-level Poisson process $\{F\}$ is Markovian; so according to the local evolution equation (13), the coupled process $\{F(x),\tau(x)\}$ is still Markovian. Let us define the joint laws

$$P_0(\tau,x)d\tau = \text{Prob}\Big\{F(x)=\phi_0 \text{ and } \tau(x,b)\in[\tau,\tau+d\tau]\Big\}$$

$$P_1(\tau,x)d\tau = \text{Prob}\Big\{F(x) = -\phi_1 \text{ and } \tau(x,b) \in [\tau, \tau + d\tau]\Big\} \tag{16}$$

They evolve according to the two coupled Master equations

$$\left\{ \begin{array}{ll} -\dfrac{\partial P_0}{\partial x} & = -\dfrac{\partial}{\partial \tau}\big[(1-\beta\phi_0\tau)P_0\big] - n_0P_0 + n_1P_1 \\ -\dfrac{\partial P_1}{\partial x} & = -\dfrac{\partial}{\partial \tau}\big[(1+\beta\phi_1\tau)P_1\big] + n_0P_0 - n_1P_1 \end{array} \right\} \tag{17}$$

Stationary distributions $P_0(\tau)$ et $P_1(\tau)$ in the limit $x \to -\infty$ are therefore the solutions of the system

$$\left\{ \begin{array}{l} \dfrac{d}{d\tau}\big[(\beta\phi_0\tau - 1)P_0\big] - n_0P_0 + n_1P_1 = 0 \\ \dfrac{d}{d\tau}\big[(\beta\phi_1\tau + 1)P_1\big] - n_0P_0 + n_1P_1 = 0 \end{array} \right\} \tag{18}$$

respectively normalized on the interval $[\frac{1}{\beta\phi_0}, +\infty]$ by

$$\int_{\frac{1}{\beta\phi_0}}^{\infty} P_0(\tau)d\tau = \frac{n_1}{n_0+n_1} \quad \text{and} \quad \int_{\frac{1}{\beta\phi_0}}^{\infty} P_1(\tau)d\tau = \frac{n_0}{n_0+n_1} \tag{19}$$

It is convenient to set $\tau_0 = \dfrac{1}{\beta\phi_0}$, $\tau_1 = \dfrac{1}{\beta\phi_1}$, $\nu_0 = n_0\tau_0$ et $\nu_1 = n_1\tau_1$, and

$$\nu = \nu_1 - \nu_0 = \frac{1}{\beta}\left(\frac{n_1}{\phi_1} - \frac{n_0}{\phi_0}\right) > 0 \tag{20}$$

which is strictly positive according to the hypothesis $F_0 > 0$ (14). The solutions of (18) (19) read, using Heaviside function θ ,

$$\left\{ \begin{array}{lll} P_0(\tau) = A\ \theta(\tau-\tau_0)\dfrac{(\tau-\tau_0)^{\nu_0-1}}{(\tau+\tau_1)^{\nu_1}} & \text{where} & A = \frac{n_1}{n_0+n_1}\frac{\Gamma(\nu_1)}{\Gamma(\nu_0)\Gamma(\nu_1-\nu_0)}(\tau_0+\tau_1)^{\nu_1-\nu_0} \\ P_1(\tau) = B\ \theta(\tau-\tau_0)\dfrac{(\tau-\tau_0)^{\nu_0}}{(\tau+\tau_1)^{\nu_1+1}} & \text{where} & B = \frac{n_0}{n_0+n_1}\frac{\Gamma(\nu_1+1)}{\Gamma(\nu_0+1)\Gamma(\nu_1-\nu_0)}(\tau_0+\tau_1)^{\nu_1-\nu_0} \end{array} \right\} \tag{21}$$

Moments of order k of these distributions diverge as soon as $k \geq \nu = (\nu_1 - \nu_0)$. They read otherwise in polynomial forms as

$$\left\{ \begin{array}{llll} \int_0^\infty d\tau\ \tau^k\ P_0(\tau) & = \dfrac{n_1}{n_0+n_1} & \tau_0^k & F\left(-k, \nu_0, 1+\nu_0-\nu_1; 1+\dfrac{\tau_1}{\tau_0}\right) \\ \int_0^\infty d\tau\ \tau^k\ P_1(\tau) & = \dfrac{n_0}{n_0+n_1} & \tau_0^k & F\left(-k, \nu_0+1, 1+\nu_0-\nu_1; 1+\dfrac{\tau_1}{\tau_0}\right) \end{array} \right\} \tag{22}$$

The velocity can be directly deduced from first moment of τ_∞ (7)

$$\frac{1}{V} = \frac{1}{D_0} < \tau_\infty >= \frac{1}{D_0}\int_0^\infty d\tau\ \tau\ \big[P_0(\tau) + P_1(\tau)\big] \tag{23}$$

There exists therefore a dynamical phase transition at $\nu = (\nu_1 - \nu_0) = 1$

$$V = \left\{ \begin{array}{lll} 0 & & \text{if} \quad \nu \leq 1 \\ \dfrac{D_0\,(\nu - 1)}{\dfrac{n_1}{n_0+n_1}\left[\tau_0(\nu_1 - 1) + \tau_1\nu_0\right] + \dfrac{n_0}{n_0+n_1}\left[\tau_0\nu_1 + \tau_1(\nu_0 + 1)\right]} & & \text{if} \quad \nu \geq 1 \end{array} \right\} \tag{24}$$

More generally, the algebraic decay of the distributions $P_{0,1}(\tau)$ in the limit $\tau \to \infty$

$$P_{0,1}(\tau) \underset{\tau\to\infty}{\propto} \frac{1}{\tau^{1+\nu}} \qquad \text{with} \qquad \nu = \nu_1 - \nu_0 = \frac{1}{\beta}\left(\frac{n_1}{\phi_1} - \frac{n_0}{\phi_0}\right) \tag{25}$$

will generate the same succession of anomalous diffusion behaviours for the cumulants of the Brownian particle position as a function of the parameter ν, as the one that appears in the white noise case as a function of the parameter μ [3]. In particular, the thermal average of the position of the particle will grow linearly in time only in the finite velocity phase

$$\overline{x(t)} \equiv \int_{-\infty}^{+\infty} dx\; x\; P(x,t\,|x_0,0) \underset{t\to\infty}{\propto} V\; t \qquad \text{if} \;\; \nu > 1 \tag{26}$$

However, in the vanishing velocity phase, the thermal average of the position will grow slower than linearly, as a power of time with the exponant ν which depends continuously on the parameters of the disorder

$$\overline{x(t)} \equiv \int_{-\infty}^{+\infty} dx\; x\; P(x,t\,|x_0,0) \underset{t\to\infty}{\propto} t^{\nu} \qquad \text{if} \;\; 0 < \nu < 1 \tag{27}$$

Let us now consider the stationary flux J_∞ going through a sample between two concentrations $P_0 > 0$ and $P_N = 0$ in the limit $N \to \infty$. The change of variable (10) gives immediately the two joint laws $\mathcal{P}_0(J)$ et $\mathcal{P}_1(J)$ from result (21), with the notations $J_0 = \frac{D_0 P_0}{\tau_0}$ and $J_1 = \frac{D_0 P_0}{\tau_1}$

$$\left\{ \begin{array}{l} \mathcal{P}_0(J) = \mathcal{A}\;\theta(J)\;\theta(J_0 - J)\;J^{\nu_1-\nu_0-1}\;\dfrac{(J_0 - J)^{\nu_0-1}}{(J_1 + J)^{\nu_1}} \\ \\ \mathcal{P}_1(J) = \mathcal{B}\;\theta(J)\;\theta(J_0 - J)\;J^{\nu_1-\nu_0-1}\;\dfrac{(J_0 - J)^{\nu_0}}{(J_1 + J)^{\nu_1+1}} \end{array} \right\} \tag{28}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two normalization constants.
These two distributions for the flux are concentrated on the bounded interval $[0, J_0]$. The transition at $\nu \equiv \nu_1 - \nu_0 = 1$ for the self-averaging velocity (24) corresponds for the flux probability distributions to a transition of the behaviour in the limit $J \to 0$

$$\mathcal{P}_{0,1}(J) \underset{J\to 0}{\propto} J^{\nu-1} \longrightarrow \left\{ \begin{array}{lll} +\infty & \text{if} & \nu < 1 \\ \\ 0 & \text{if} & \nu > 1 \end{array} \right\} \tag{29}$$

The vanishing velocity phase $\nu = \nu_1 - \nu_0 < 1$ is characterised by the divergence of $\mathcal{P}_0$ and $\mathcal{P}_1$ at the origin $J \to 0$. On the contrary in the finite velocity phase $\nu > 1$, the probability distributions of the flux vanish at the origin.

Some curves $\mathcal{P}_{0,1}(J)$ are drawn on Figures 2 and 3 for different values of the parameters. They illustrate in particular the transitions at $\nu = 1$ we just mentioned (29) and the transion at $\nu_0 = 1$ for the behaviour of $\mathcal{P}_0(J)$ as $J \to J_0$. We refer to the figures in [8] for comparison with the uncorrelated case.

In this article, we have analysed some properties of anomalous diffusion in a random medium described by a quenched random force $\{F\}$ distributed as a two-level Poisson process. In particular, we showed that there exists a dimensionless parameter ν, which governs the asymptotic behaviours of the probability distributions $P_{0,1}(\tau)$ in the limit $\tau \to \infty$. As a result, this parameter ν also controlls the dynamical phase transition for the velocity and the behaviours of the probability distributions $\mathcal{P}_{0,1}(J)$ of the flux in the limit $J \to 0$. All this is qualitatively the same as what is known in the white noise case. However, the probability distributions for the random variable τ_∞ and for the stationary flux J_∞ are very different from the white noise case outside the asymptotic regimes $\tau \to \infty$ and $J \to 0$. Indeed, they present restricted supports, as a consequence of the bounded character of the random force, and they are given in terms of only rational functions and no exponential.

More generally, the two-level Poisson process we considered is a technically very convenient disorder model, since it allows analytical studies despite the presence of correlations. It has already been used in various contexts, like one-dimensional quantum localization [10] [11], non-linear systems coupled to a random environment [12], and noise-induced perturbations on Josephson junctions [13]. The limitation to two-level processes simplifies computations, but the approach can be generalized to any finite number of levels for the random process [14].

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Figures captions

1) Example of the random potential $U(x)$ seen by the Brownian particle when the random force $\{F\}$ is a two-level Poisson process.

2) Examples of flux probability distributions $\mathcal{P}_0(J)$ (plain line) and $\mathcal{P}_1(J)$ (dashed line) in the vanishing velocity phase $0 < \nu < 1$ for
   2a) $\nu_0 < 1$
   2b) $\nu_0 > 1$

3) Examples of flux probability distributions $\mathcal{P}_0(J)$ (plain line) and $\mathcal{P}_1(J)$ (dashed line) in the finite velocity phase $\nu > 1$ for
   3a) $\nu_0 < 1$
   3b) $\nu_0 > 1$